CHIBA UNIV. Preprint

CHIBA-EP-117
hep-th/9911242
November 1999

Non-Abelian Stokes Theorem and Quark Confinement in SU(N) Yang-Mills Gauge Theory

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Abstract

We derive a new version of non-Abelian Stokes theorem for the Wilson loop in the $SU(N)$ case by making use of the coherent state representation on the coset space $SU(N)/U(1)^{N-1} = F_{N-1}$, the flag space. We consider the $SU(N)$ Yang-Mills theory in the maximal Abelian gauge in which $SU(N)$ is broken down to $U(1)^{N-1}$. First, we show that the Abelian dominance in the string tension follows from this theorem and the Abelian-projected effective gauge theory which was derived by one of the authors. Next (but independently), combining the non-Abelian Stokes theorem with a novel reformulation of the Yang-Mills theory recently proposed by one of the authors, we proceed to derive the area law of the Wilson loop in four-dimensional $SU(N)$ Yang-Mills theory in the maximal Abelian gauge. Due to dimensional reduction, the planar Wilson loop in the fundamental representation in four-dimensional $SU(N)$ Yang-Mills theory can be estimated by the diagonal (Abelian) Wilson loop defined in the two-dimensional CP^{N-1} model. This derivation shows that the fundamental quarks are confined by a single species of magnetic monopole. The non-Abelian Stokes theorem extracts the gauge-invariant magnetic monopole contribution and gives the $SU(N)$ generalization of the 't Hooft-Polyakov tensor for the magnetic monopole. An origin of the area law is related to the geometric phase of the Wilczek-Zee holonomy for $U(N-1)$. The calculations are performed by the instanton calculus (in the dilute instanton-gas approximation) or by the large N expansion (in the leading order).

Key words: quark confinement, topological field theory, magnetic monopole, non-Abelian Stokes theorem, topological soliton
PACS: 12.38.Aw, 12.38.Lg

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Contents

1 Introduction

In the mid-1960s, Gell-Mann and Zweig [1] proposed a model in which all the hadrons (i.e., baryons and mesons) are composed by the fundamental constituent having fractional charges, $(1/3)e$ or $(2/3)e$ with e being the elementary charge. The predictions extracted from this model have been consistent with the results of various experiments performed over the past thirty years. Now the fundamental constituent is called the quark and hence the proposed theory is called the quark model. It is widely believed that the strong interaction among quarks or anti-quarks is mediated by the gluon which is described by the quantized Yang-Mills gauge field theory [2]. Therefore, the fundamental theory describing the quark and the gluon is given by the quantum chromodynamics (QCD) which is a non-Abelian gauge theory with the gauge group $G = SU(3)$ corresponding to three colors. However, an isolated quark and an isolated anti-quark have never been observed in experiments. Nowadays they are considered to be confined in the hadrons. This is the hypothesis of quark confinement. Quark confinement could be explained theoretically within the framework of QCD, although no one has achieved a rigorous proof of quark confinement until now. This is one of the most principal problems to be resolved in theoretical physics.

QCD has a remarkable property called the asymptotic freedom which was discovered by Gross, Wilczek and Politzer and independently by 't Hooft [3]. The asymptotic freedom does not show up in the most successful quantized field theory, quantum electrodynamics (QED). As is well known, QED is the Abelian gauge theory for the electron and the photon where the electromagnetic interaction is described by the quantized Maxwell gauge field theory with a gauge group $G = U(1)$. The asymptotic freedom is a consequence of gluon self-interactions. Therefore, the asymptotic freedom is a very characteristic feature of non-Abelian gauge theory.

The purpose of this article is to demonstrate quark confinement within QCD based on the Wilson criterion for quark confinement [4], i.e., area law of the Wilson loop. The Wilson loop is a gauge invariant quantity and hence the Wilson criterion is also gauge invariant. The formulation of lattice gauge theory proposed by Wilson [4] is manifestly gauge invariant and does not need the gauge fixing. It is easy to show that the strong coupling expansion in the lattice gauge theory leads to the area law of the Wilson loop. However, this result could not be continued to the weak coupling region where the string tension is expected to obey the scaling law suggested from the result of the renormalization group based on the loop calculations. The first indication of area law of the Wilson loop for arbitrary coupling constant was given by Creutz [5] based on numerical simulations within the lattice gauge theory by for $G = SU(2)$ and $SU(3)$. Although the numerical evidence of quark confinement was indeed a great progress towards the complete understanding of quark confinement, the analytical proof is still lacking.

This work was initiated to justify the dual superconductor picture of QCD vacuum proposed in the mid-1970s [6] within the framework of the continuum quantum field theory. For the dual superconductivity to occur, magnetic monopoles must be condensed, just as the ordinary superconductivity needs condensation of Cooper pairs. In fact, the importance and the validity of taking into account magnetic monopoles in quark confinement has been demonstrated at least for the simplified four-dimensional and lower dimensional models, especially, by Polyakov [7], and recently for the four-dimensional Yang-Mills theory and QCD with extended supersymmetries by Seiberg and Witten [8]. In this scenario quark confinement is realized due to condensation of magnetic monopoles. Recent developments of numerical simulations on the lattice [9] confirm the dual superconductivity in QCD, at least, under a specific gauge fixing called the Abelian gauge [10].

This article gives a detailed exposition of the results on quark confinement which were announced in the previous article [11] for $G = SU(3)$ together with new results for $G = SU(N)$ with arbitrary N. They are extensions of the analyses of the Yang-Mills theory in the maximal Abelian (MA) gauge given in a series of articles [12, 13, 14, 15, 16, 17] where the case of $SU(2)$ was explicitly worked out. In this process, we have found that the extension from $SU(2)$ to $SU(3)$ is non-trivial, but $SU(3)$ to $SU(N)$, $N > 3$ is rather straightforward. New features come out when we begin to analyze the $SU(N)$ case $N \geq 3$. It seems that they have been overlooked so far in the conventional approach based on the maximal Abelian gauge.

The MA gauge is a partial gauge fixing from the original non-Abelian gauge group G to its subgroup H [10] in which the gauge degrees of freedom of the coset G/H is fixed. Even after the MA gauge, there is a residual gauge group H which is taken to be the maximal torus subgroup $H = U(1)^{N-1}$. After the MA gauge, the magnetic monopole is expected to occur, since the Homotopy group $\pi_2(G/H)$ is non-trivial, i.e.,

$$\pi_2(SU(N)/U(1)^{N-1}) = \pi_1(U(1)^{N-1}) = \mathbf{Z}^{N-1}. \tag{1.1}$$

This implies that the breaking of gauge group $G \to H$ by partial gauge fixing leads to $(N-1)$ species of magnetic monopoles. However, we do not necessarily need to take the maximal breaking $SU(N) \to U(1)^{N-1}$, although the maximal torus group is desirable as a gauge group of the low-energy effective *Abelian* gauge theory [12]. Actually, even if we restrict H to the continuous subgroup [1] of G, there is other possibilities for choosing H, i.e., we can choose a subgroup $\tilde{H}$ such that

$$G \supset \tilde{H} \supset H := U(1)^{N-1}. \tag{1.2}$$

The possible number of cases for choosing $\tilde{H}$ increases as N increases. We have found [11] that the group $\tilde{H}$ may change depending on the representation to which the quark belongs when $N \geq 3$, and that it suffices to take $\tilde{H} = U(N-1)$ for the fundamental quark to be confined in the sense of area law of the Wilson loop under the partial gauge fixing. Here $\tilde{H}$ is equal to the maximal stability group specified by the highest-weight state of the representation of the quark in the Wilson loop. This is a new feature which does not show up in the SU(2) case. Nevertheless, this does not mean that the choice of the maximal torus does not leads to quark confinement. In fact, even if we choose the maximal torus, the area law is derived. This is because the coset $G/\tilde{H}$ is contained in G/H, i.e., $G/\tilde{H} \subset G/H$, so that the Wilson loop does not feel the whole of G/H, but only feels the components of $G/\tilde{H}$ which are contained

[1]The possibility of discrete subgroup is extensively investigated recently from the viewpoint of Abelian gauge, e.g., the center Z_N for $SU(N)$, see e.g. [18].

in G/H. In other words, the variables belonging to $G/H - G/\tilde{H}$ are irrelevant for the expectation value of the Wilson loop as can be seen from the non-Abelian Stokes theorem (NAST) which was announced in [11] and is derived in this article. Therefore, single kind of magnetic monopole is sufficient for confining fundamental quark, since

$$\pi_2(SU(N)/U(N-1)) = \pi_1(U(1)) = \mathbf{Z}. \tag{1.3}$$

Our results show that two partial gauge fixings $SU(3) \to U(2)$ and $SU(3) \to U(1) \times U(1)$ lead to the same result for confinement as far as the fundamental quarks is concerned. [2]

The NAST plays the crucial role in this article. The NAST have a number of versions which have already been derived by many authors [20]. A version of NAST derived in this article is based on the idea of Dyakonov and Petrov [21] who derived a $SU(2)$ version and suggested a way of generalization. We derive a version of NAST based on the coherent state representation [22] on the flag space [23, 24], not based on the way suggested by them. The coherent state representation was already used in a different fashion to derive a $SU(2)$ version of the NAST in [15], but the extension to $SU(N)$, $N \ge 3$ was a non-trivial issue which prevented us from presenting immediate publication of general $SU(N)$ results. The NAST is not only mathematically (or technically) important, but also physically interesting as follows.

First, the NAST enables us to write the non-Abelian Wilson loop,

$$W^C[\mathcal{A}] := \frac{1}{N}\mathrm{tr}\left[\mathcal{P}\exp\left(ig\oint_C \mathcal{A}\right)\right], \tag{1.4}$$

in terms of the Abelian-field strength (curvature two-form) $f = da$ with the Abelian gauge potential (connection one-form) a,

$$W^C[\mathcal{A}] = \int [d\mu(V)]_C \exp\left(ig\oint_C a\right) = \int [d\mu(V)]_C \exp\left(ig\int_S f\right). \tag{1.5}$$

Combining this fact with the Abelian-projected effective gauge theory (APEGT) derived in [12], we can explain the Abelian dominance [25, 26] in the Wilson loop. The APEGT is an Abelian gauge theory obtained by integrating out the massive degrees of freedom, i.e., the off-diagonal gluon gauge field A_μ^a with mass m_A. Hence the APEGT is written in terms of the diagonal massless gauge field a_μ^i and the anti-symmetric (Abelian) tensor field $B_{\mu\nu}^i$ together with the ghost and anti-ghost fields $C^a, \bar{C}^a$, where the index i denotes the diagonal components and a the off-diagonal ones. Therefore the APEGT is regarded as the low-energy effective theory (LEET) which is valid in the long-distance (or low-energy) region $R > m_A^{-1}$. The Abelian gauge field b_μ^i obtained after the Hodge decomposition of $B_{\mu\nu}^i$ can be identified with the Abelian gauge field dual to a_μ^i. In fact, we can obtain the theory with an action $S[b]$ written in terms of b_μ^i alone by integrating out all the fields other than b_μ^i in APEGT and the theory can be rewritten into the dual Ginzburg-Landau theory, i.e., dual Abelian Higgs theory, provided that the magnetic monopole condensation occurs,

[2]See [19] for a result of the simulation on a lattice.

i.e., $\langle k_\mu k_\mu \rangle \neq 0$. In the dual Ginzburg-Landau theory, the coupling constant g in the original theory is replaced by the inverse coupling constant $1/g$ which is proportional to the magnetic charge. Therefore the dual theory is identified with the magnetic theory. On the other hand, the theory with an action $S[a]$ written in terms of a^i alone is an Abelian gauge theory, but the scale dependence of the coupling constant $g(\mu)$ is the same as the original Yang-Mills theory, i.e., the low-energy effective Abelian gauge theory exhibits asymptotic freedom reproducing the original renormalization-group beta function $\beta(g)$. This is a manifestation of an approximate weak-strong or electro-magnetic duality between two low-energy effective theories described by $S[a]$ and $S[b]$.

Next, the NAST is able to separate the piece ω which corresponds to the magnetic monopole in the Abelian field $a = C + \omega$. Indeed, we can write the $SU(N)$ version of the 't Hooft-Polyakov tensor describing the magnetic monopole [27]. So we can separate the contribution of the magnetic monopole in the area law of the Wilson loop and explain the magnetic monopole dominance in quark confinement. In fact, our derivation of area law estimates only the monopole contribution $\Omega_K = d\omega$. Moreover, the NAST tells us that the essential ingredient to the area law lies in the geometric phase which is concerned with the holonomy group of $U(N-1)$. Thus quark confinement is intimately related to the geometry of Yang-Mills gauge theory, in sharp contrast with the conventional wisdom.

We will present two ways to show the area law of the Wilson loop by making use of the NAST. One way is to use the APEGT for estimating the diagonal Wilson loop; for sufficiently large Wilson loop ($|C| \gg m_A^{-1}$), the expectation value of the non-Abelian Wilson loop in Yang-Mills theory is reduced to that of Abelian Wilson loop in APEGT, i.e.,

$$\left\langle W^C[\mathcal{A}] \right\rangle_{YM} = \left\langle \exp\left(ig \oint_C a \right) \right\rangle_{YM} \to \left\langle \exp\left(ig \oint_C a \right) \right\rangle_{APEGT}. \tag{1.6}$$

Then we can apply the result of [14], i.e., confinement in the *Abelian* gauge theory to show quark confinement in Yang-Mills theory.

Another way is to treat the non-Abelian gauge theory directly, without going through the effective Abelian gauge theory, based on the novel reformulation of Yang-Mills theory in the MA gauge which has been proposed by one of the authors [13]. The novel reformulation regards the Yang-Mills theory as the perturbative deformation of a topological quantum field theory (TQFT). An advantage of this reformulation in the MA gauge is that the derivation of the area law of the non-Abelian Wilson loop in the four-dimensional $SU(N)$ Yang-Mills theory is reduced to that of the diagonal (Abelian) Wilson loop in the two-dimensional coset (G/H) non-linear sigma (NLS) model, at least when the Wilson loop is planar. Therefore the four-dimensional problem is reduced to the two-dimensional issue. This dimensional reduction is a remarkable feature of the modified MA gauge [3] caused by hidden supersymmetry. The Yang-Mills coupling constant g of the four-dimensional Yang-Mills theory is

[3]We must modify the MA gauge slightly in order to keep the supersymmetry where the supersymmetry is expressed by orthosymplectic group $OSp(4/2)$.

traded into the coupling constant in two-dimensional NLS model. Hence the coupling constant is expected to run in the same way as the original Yang-Mills theory, since the perturbative deformation part provides the necessary running as is well known by the loop calculation. For the fundamental quark, we are allowed to restrict the flag space $F_{N-1} := SU(N)/U(1)^{N-1}$ to the complex projective space $CP^{N-1} := SU(N)/U(N-1)$. This greatly simplifies the actual treatment.

Another advantage of this reduction is that the magnetic monopole contribution to the Wilson loop in the four-dimensional Yang-Mills theory in the MA gauge is shown to be equal to the instanton contribution in the corresponding two-dimensional NLS model. Indeed, the diagonal Wilson loop is written as the area integral of the instanton density Ω_K over the area S bounded by the loop C. This correspondence may shed more light on the strong correlation between magnetic monopoles and instantons observed in the Monte Carlo simulations, since the two-dimensional instanton is identified as a subclass of the four-dimensional Yang-Mills instanton, see e.g. [13].

In this article the expectation value of the Wilson loop is estimated by combining the instanton calculus and the large N expansion, see [28, 29, 30, 31] for reviews of large N expansion. We focus on the CP^{N-1} model corresponding to the fundamental quark. First, the instanton calculus is performed within the dilute gas approximation. It is shown that the calculation in SU(N) case is reduced to the SU(2) case. It is well known that the large N expansion is a non-perturbative technique which can be systematically improved. We derive the area law in the leading order of the large N expansion, namely, in the region of large N and weak coupling g. We hope that our derivation of quark confinement based on the dimensional reduction and the large N expansion may shed more light on the relationship between QCD and string theory, as first suggested by 't Hooft [32].

This article is organized as follows. In the former half, we give a derivation of the NAST and discuss its implications. Section 2 and 3 are preparations for section 4. In section 2, we review how to construct the coherent state on the flag space for the general compact semi-simple group G. In section 3, we present explicit form of the coherent state on the flag space for $G = SU(N)$. We define the maximal stability group $\tilde{H}$ which is very important in the following discussions. In section 4, making use of the results of section 2 and 3, we derive a new version of non-Abelian Stokes theorem for $G = SU(N)$. Although we discuss only the case of $SU(N)$ explicitly, it is straightforward to extend this theorem to arbitrary compact semi-simple group G. This version of non-Abelian Stokes theorem is very interesting not only from the mathematical but also from the physical point of view, since the non-Abelian Wilson loop is expressed as the surface integral of the two-form (i.e., the generalized 't Hooft-Polyakov tensor) which leads to the magnetic monopole. This fact is intimately related with the Abelian and magnetic monopole dominance in quark confinement as discussed in subsequent sections.

In the latter half, we derive area law of the Wilson loop. In section 5, we discuss the magnetic monopole in $SU(N)$ Yang-Mills theory. In order to specify the type of possible magnetic monopoles, it turns out that the maximal stability group is more important than the maximal torus group H. In section 6, Abelian dominance in the Wilson loop is shown in the $SU(N)$ Yang-Mills theory in the maximal Abelian gauge based on the Abelian-projected effective gauge theory and the non-Abelian Stokes theorem. In section 7, we briefly review a novel reformulation of the Yang-Mills theory which has been proposed by one of the authors [13] to derive quark confinement. This reformulation was called the (perturbative) deformation of the topological quantum field theory. We apply this reformulation to derive area law of the Wilson loop in $SU(N)$ Yang-Mills theory in section 8 and 9. In section 9, we show within this reformulation that the area law of the Abelian Wilson loop in the two-dimensional nonlinear sigma model for the flag space $G/\tilde{H}$ is sufficient to derive the area law of the four-dimensional Yang-Mills theory in the maximal Abelian gauge. At the same time, this derivation leads to the magnetic monopole dominance in the area law. In section 8, we demonstrate the area law of the Wilson loop in the nonlinear sigma model based on the naive instanton calculus. For the fundamental quark, we have only to deal with the CP^{N-1} model. In section 9, we derive the area law based on the large N expansion. These results imply the area law of the non-Abelian Wilson loop in the four-dimensional $SU(N)$ Yang-Mills theory. The final section is the conclusion of this article.

In Appendix A, we give derivations of the inner product of the coherent states and the invariant measure on the flag space which are presented in section 3. In Appendix B, we explain how to obtain CP^1 and CP^2 by gluing the complex planes. In Appendix C, we explain two ways to characterize the element of the flag space and how to write the NLS model using these parameterizations. In Appendix D, we summarize the large N expansion of CP^{N-1} model. In Appendix E, supplementary materials on the $1/N$ expansion are presented.

2 Coherent state and maximal stability group

First of all, we construct the *coherent state* $|\xi, \Lambda\rangle$ corresponding to the coset representatives $\xi \in G/\tilde{H}$. We follow the method of Feng, Gilmore and Zhang [22]. For inputs, we prepare

(a) the gauge group [4] G and the Lie algebra $\mathcal{G}$ of G with the generators $\{T^A\}$ which obey the commutation relations

$$[T^A, T^B] = if^{AB}{}_C T^C, \tag{2.1}$$

where $f^{AB}{}_C$ are the structure constants of the Lie algebra. If the Lie algebra is semi-simple, it is more convenient to rewrite the Lie algebra in terms of the Cartan basis $\{H_i, E_\alpha, E_{-\alpha}\}$. There are two types of basic operators in the Cartan basis, H_i and E_α. The operators H_i may be taken as diagonal, while E_α are the off-diagonal shift operators. They obey the commutation relations,

$$[H_i, H_j] = 0, \tag{2.2}$$

[4]Note that any compact semi-simple Lie group is a direct product of compact simple Lie group. Therefore, it is sufficient to consider the case of a compact simple Lie group. In the following we assume that G is a compact simple Lie group, i.e., a compact Lie group with no closed connected invariant subgroup.

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \tag{2.3}$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i, \tag{2.4}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha;\beta} E_{\alpha+\beta} & (\alpha+\beta \in R) \\ 0 & (\alpha+\beta \notin R, \alpha+\beta \neq 0) \end{cases}, \tag{2.5}$$

where R is the root system, i.e., a set of root vectors $\{\alpha_1, \cdots, \alpha_r\}$ with r the ranks of G;

(b) the Hilbert space V^Λ as a carrier (the representation space) of the unitary irreducible representation Γ^Λ of G;

(c) a reference state $|\Lambda\rangle$ within the Hilbert space V^Λ, which can be normalized to unity, $\langle\Lambda|\Lambda\rangle = 1$.

We define the *maximal stability* subgroup (*isotropy* subgroup) $\tilde{H}$ as a subgroup of G that consists of all the group elements h that leave the reference state $|\Lambda\rangle$ invariant up to a phase factor, i.e.,

$$h|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}, h \in \tilde{H}. \tag{2.6}$$

The phase factor is unimportant in the following discussion because we consider the expectation value of any operators in the coherent state. Let H be the Cartan subgroup of G, i.e., the maximal commutative semi-simple subgroup in G and $\mathcal{H}$ be the Cartan subalgebra in $\mathcal{G}$, i.e., the Lie algebra for group H. The maximal stability subgroup $\tilde{H}$ includes the Cartan subgroup $H = U(1)^r$, i.e., $H \subset \tilde{H}$.

For every element $g \in G$, there is a unique decomposition of g into a product of two group elements,

$$g = \xi h, \quad \xi \in G/\tilde{H}, \quad h \in \tilde{H}, \tag{2.7}$$

for $g \in G$. We can obtain a unique coset space $G/\tilde{H}$ for a given $|\Lambda\rangle$. The action of arbitrary group element $g \in G$ on $|\Lambda\rangle$ is given by

$$g|\Lambda\rangle = \xi h|\Lambda\rangle = \xi|\Lambda\rangle e^{i\phi(h)}. \tag{2.8}$$

The coherent state is constructed as $|\xi, \Lambda\rangle = \xi|\Lambda\rangle$. This definition of the coherent state is in one-to-one correspondence with the coset space $G/\tilde{H}$ and the coherent states preserve all the algebraic and topological properties of the coset space $G/\tilde{H}$.

If $\Gamma^\Lambda(\mathcal{G})$ is Hermitian, then $H_i^\dagger = H_i$, and $E_\alpha^\dagger = E_{-\alpha}$. Every group element $g \in G$ can be written as the exponential of a complex linear combination of diagonal operators H_i and off-diagonal shift operators E_α. Let $|\Lambda\rangle$ be the highest-weight state, i.e., $H_j|\Lambda\rangle = \Lambda_j|\Lambda\rangle$, $E_\alpha|\Lambda\rangle = 0$ for $\alpha \in R_+$, where $R_+(R_-)$ is a subsystem of positive (negative) roots. Thus the coherent state is given by [22]

$$|\xi, \Lambda\rangle = \xi|\Lambda\rangle = \exp\left[\sum_{\beta \in R_-} \left(\eta_\beta E_\beta - \bar{\eta}_\beta E_\beta^\dagger\right)\right]|\Lambda\rangle, \quad \eta_\beta \in \mathbf{C}, \tag{2.9}$$

such that

(i) $|\Lambda\rangle$ is annihilated by all the (off-diagonal) shift-up operators E_α with $\alpha \in R_+$, $E_\alpha|\Lambda\rangle = 0(\alpha \in R_+)$;

(ii) $|\Lambda\rangle$ is mapped into itself by all diagonal operators H_i, $H_i|\Lambda\rangle = \Lambda_i|\Lambda\rangle$;

(iii) $|\Lambda\rangle$ is annihilated by some shift-down operators E_α with $\alpha \in R_-$, not by other E_β with $\beta \in R_-$: $E_\alpha|\Lambda\rangle = 0(\text{some } \alpha \in R_-)$; $E_\beta|\Lambda\rangle = |\Lambda + \beta\rangle(\text{some } \beta \in R_-)$;

and the sum $\sum_\beta$ is restricted to those shift operators E_β which obey (iii).

The coherent states are normalized to unity,

$$\langle \xi, \Lambda | \xi, \Lambda \rangle = 1. \tag{2.10}$$

The coherent state spans the entire space V^Λ. However, the coherent state are non-orthogonal,

$$\langle \xi', \Lambda | \xi, \Lambda \rangle \neq 0. \tag{2.11}$$

By making use of the the group-invariant measure $d\mu(\xi)$ of G which is appropriately normalized, we obtain

$$\int |\xi, \Lambda\rangle d\mu(\xi) \langle \xi, \Lambda| = I, \tag{2.12}$$

which shows that the coherent states are complete, but overcomplete. This resolution of identity is very important to obtain the path integral formula of the Wilson loop given in section 4.

The coherent states $|\xi, \Lambda\rangle$ are in one-to-one correspondence with the coset representatives $\xi \in G/\tilde{H}$,

$$|\xi, \Lambda\rangle \leftrightarrow G/\tilde{H}. \tag{2.13}$$

In other words, $|\xi, \Lambda\rangle$ and $\xi \in G/\tilde{H}$ are topologically equivalent.

3 Flag space and coherent state for $SU(N)$

3.1 $SU(2)$ coherent state

In the case of SU(2), the maximal stability group agrees with the maximal torus group $U(1)$ irrespective of the representation. The SU(2) case is well known, see e.g. [15]. The weight and root diagrams are given in Fig.1.

The coherent state for $F_1 := SU(2)/U(1)$ is obtained as

$$|j, w\rangle = \xi(w)|j, -j\rangle = e^{\zeta J_+ - \bar{\zeta} J_-}|j, -j\rangle = \frac{1}{(1 + |w|^2)^j} e^{w J_+}|j, -j\rangle, \tag{3.1}$$

where $|j, -j\rangle$ is the lowest state, $|j, m = -j\rangle$ of $|j, m\rangle$, and

$$J_+ = J_1 + iJ_2, \quad J_- = J_+^\dagger, \quad w = \frac{\zeta \sin|\zeta|}{|\zeta| \cos|\zeta|}. \tag{3.2}$$

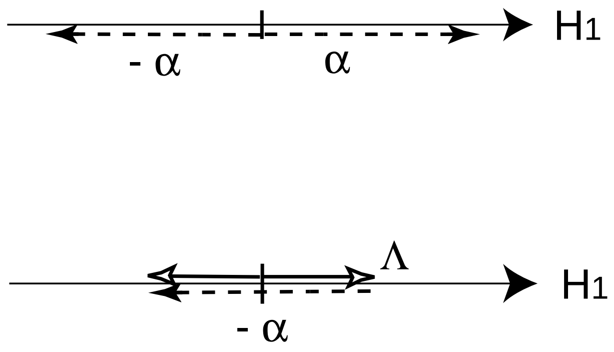

Figure 1: Root diagram and weight diagram of the fundamental representation of SU(2) where Λ is the highest weight of the fundamental representation.

Note that $(1+|w|^2)^{-j}$ is a normalization factor to ensure $\langle j,w|j,w\rangle = 1$, which is obtained from the Baker-Campbell-Hausdorff (BCH) formulas. The invariant measure is given by

$$d\mu = \frac{2j+1}{4\pi}\frac{dwd\bar{w}}{(1+|w|^2)^2}. \tag{3.3}$$

For $J_A = \frac{1}{2}\sigma^A (A=1,2,3)$ with Pauli matrices σ^A, we obtain $J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and

$$e^{wJ_+} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in F_1 = CP^1 = SU(2)/U(1) \cong S^2. \tag{3.4}$$

The complex variable w is a CP^1 variable written as $w = e^{i\phi}\cot\frac{\theta}{2}$, in terms of the polar coordinate on S^2 or Euler angles, see [13]. We can introduce the $O(3)$ vector $\mathbf{n}$,

$$n^1 := \sin\theta\cos\phi, \quad n^2 := \sin\theta\sin\phi, \quad n^3 := \cos\phi. \tag{3.5}$$

The relation

$$n^A(x) = \bar{\phi}_a(x)\sigma^A_{ab}\phi_b(x) \quad (a,b=1,2), \tag{3.6}$$

is equivalent to

$$n_1 = 2Re(\phi_1\bar{\phi}_2), \quad n_2 = 2Im(\phi_1\bar{\phi}_2), \quad n_3 = |\phi_1|^2 - |\phi_2|^2. \tag{3.7}$$

The complex coordinate w obtained by the stereographic projection from the north pole is nothing but the inhomogeneous local coordinates of CP^1 when $\phi_2 \neq 0$,

$$w = w^{(1)} + iw^{(2)} = \frac{n_1 + in_2}{1-n_3} = \frac{2\phi_1\bar{\phi}_2}{(|\phi_1|^2+|\phi_2|^2) - (|\phi_1|^2 - |\phi_2|^2)} = \frac{\phi_1}{\phi_2}. \tag{3.8}$$

The stereographic projection from the south pole leads to

$$w = \frac{n_1 + in_2}{1+n_3} = (\frac{\phi_2}{\phi_1})^*, \tag{3.9}$$

if $\phi_1 \neq 0$. The variable w is U(1) gauge invariant. Another representation of $\mathbf{n}$ is obtained by using the parameterization (3.4) of F_1 variable ξ.

$$n^A = \langle \Lambda | \xi(w)^\dagger \sigma^A \xi(w) | \Lambda \rangle = \begin{pmatrix} \bar{\phi}_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \bar{w} \\ 0 & 1 \end{pmatrix} \sigma^A \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}, \tag{3.10}$$

which leads to

$$n_1 = |\phi_1|^2 (w + \bar{w}), \quad n_2 = -i|\phi_1|^2 (w - \bar{w}), \quad n_3 = |\phi_1|^2 (1 - w\bar{w}). \tag{3.11}$$

Indeed this agrees with (3.7) if $w = (\frac{\phi_2}{\phi_1})^*$. The whole space of F_1 is covered by two charts,

$$CP^1 = U_1 \cup U_2, \quad U_a = \{(\phi_1, \phi_2) \in CP^1; \phi_a \neq 0\}. \tag{3.12}$$

3.2 $SU(3)$ coherent state

For concreteness, we first focus on the SU(3) case. The general $SU(N)$ case will be discussed in the final part of this section. The highest weight Λ of the representation specified by the Dynkin index $[m, n]$ (m, n: integers) can be written as

$$\vec{\Lambda} = m\vec{h}_1 + n\vec{h}_2, \tag{3.13}$$

where m, n are non-negative integers for the highest weight and h_1, h_2 are highest weights of two fundamental representations of SU(3) corresponding to $[1, 0], [0, 1]$ respectively (See Fig.2), i.e.,

$$\vec{h}_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \quad \vec{h}_2 = \left(0, \frac{1}{\sqrt{3}} \right). \tag{3.14}$$

Therefore, we obtain

$$\vec{\Lambda} = \left(\frac{m}{2}, \frac{m + 2n}{2\sqrt{3}} \right). \tag{3.15}$$

The generators of SU(3) in the Cartan basis are written as $\{H_1, H_2, E_\alpha, E_\beta, E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}, E_{-\alpha-\beta}\}$ where α, β are the two simple roots. See Fig.3 for the explicit choice.

If $mn = 0$, (i.e., $m = 0$ or $n = 0$), the maximal stability group $\tilde{H}$ is given by $\tilde{H} = U(2)$ with generators $\{H_1, H_2, E_\beta, E_{-\beta}\}$ (case (I)). Such a degenerate case occurs when the highest-weight vector $\vec{\Lambda}$ is orthogonal to some root vectors. See Fig.2.

If $mn \neq 0$, i.e., $m \neq 0$ and $n \neq 0$, H is the maximal torus group $\tilde{H} = U(1) \times U(1)$ with generators $\{H_1, H_2\}$ (case (II)). This is a non-degenerate case. See Fig.4.

Therefore, for the highest weight Λ in the case (I), the coset $G/\tilde{H}$ is given by

$$SU(3)/U(2) = SU(3)/(SU(2) \times U(1)) = CP^2, \tag{3.16}$$

whereas in the case (II)

$$SU(3)/(U(1) \times U(1)) = F_2. \tag{3.17}$$

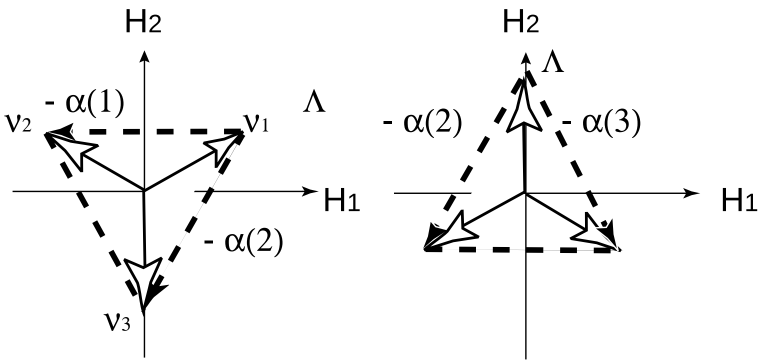

Figure 2: The weight diagram and root vectors required for defining the coherent state in the fundamental representations $[1,0] = \mathbf{3}$, $[0,1] = \mathbf{3}^*$ of SU(3) where $\vec{\Lambda} = \vec{h}_1 = \nu^1 := (\frac{1}{2}, \frac{1}{2\sqrt{3}})$ is the highest weight of the fundamental representation and other weight are $\nu^2 := (-\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and $\nu^3 := (0, -\frac{1}{\sqrt{3}})$.

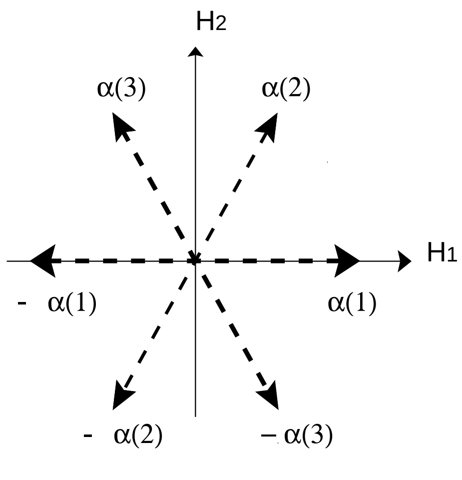

Figure 3: The root diagram of SU(3) where positive root vectors are given by $\alpha^{(1)} = (1,0)$, $\alpha^{(2)} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\alpha^{(3)} = (\frac{-1}{2}, \frac{\sqrt{3}}{2})$. Here we have used the same weight ordering as SU(N) case (see (3.45)) defining the simple roots. Then the two simple roots are given by $\alpha^1 := \alpha^{(1)}$, $\alpha^2 := \alpha^{(3)}$.

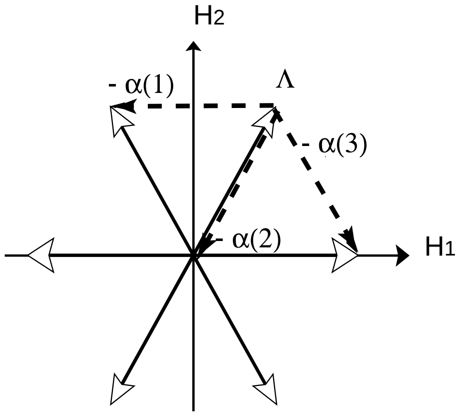

Figure 4: The weight diagram and root vectors required for defining the coherent state in the adjoint representation $[1,1] = \mathbf{8}$ of SU(3) where $\Lambda = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ is the highest weight of the adjoint representation.

Here, CP^n is the complex projective space and F_n is the flag space [23]. Therefore, the two fundamental representations belong to the case (I), so the maximal stability group is $U(2)$, rather than the maximal torus group $U(1) \times U(1)$. The implications of this fact to the mechanism of quark confinement is discussed in the subsequent sections.

The coherent state for $F_2 = SU(3)/U(1)^2$ is given by

$$|\xi, \Lambda\rangle = \xi(w)|\Lambda\rangle := V^\dagger(w)|\Lambda\rangle, \tag{3.18}$$

with the highest(lowest)-weight state $|\Lambda\rangle$, i.e.,

$$|\xi, \Lambda\rangle = \exp\left[\sum_{\alpha\in R_+}(\zeta_\alpha E_{-\alpha} - \bar{\zeta}_\alpha E_{-\alpha}^\dagger)\right]|\Lambda\rangle = e^{-\frac{1}{2}K(w,\bar{w})}\exp\left[\sum_{\alpha\in R_+}\tau_\alpha E_{-\alpha}\right]|\Lambda\rangle, \tag{3.19}$$

where $e^{-\frac{1}{2}K}$ is the normalization factor obtained from the Kähler potential (explained later)

$$K(w,\bar{w}) := \ln[(\Delta_1(w,\bar{w}))^m(\Delta_2(w,\bar{w}))^n], \tag{3.20}$$
$$\Delta_1(w,\bar{w}) := 1 + |w_1|^2 + |w_2|^2, \quad \Delta_2(w,\bar{w}) := 1 + |w_3|^2 + |w_2 - w_1 w_3|^2. \tag{3.21}$$

The coherent state $|\xi, \Lambda\rangle$ is normalized, so that $\langle\xi, \Lambda|\xi, \Lambda\rangle = 1$. We show in Appendix A that the inner product is given by

$$\langle\xi', \Lambda|\xi, \Lambda\rangle = e^{K(w,\bar{w}')}e^{-\frac{1}{2}[K(w',\bar{w}')+K(w,\bar{w})]}, \tag{3.22}$$

where

$$K(w,\bar{w}') := \ln[1 + \bar{w}_1{}'w_1 + \bar{w}_2{}'w_2]^m[1 + \bar{w}_3{}'w_3 + (\bar{w}_2{}' - \bar{w}_1{}'\bar{w}_3{}')(w_2 - w_1 w_3)]^n. \tag{3.23}$$

Note that $K(w,\bar{w}')$ reduces to the Kähler potential $K(w,\bar{w})$ when $w'=w$, in agreement with the normalization. It follows from the general formula (see $SU(N)$ case) that the $SU(3)$ invariant measure is given (up to a constant factor) by

$$d\mu(\xi) = d\mu(w,\bar{w}) = D(m,n)[(\Delta_1)^m(\Delta_2)^n]^{-2}\prod_{\alpha=1}^{3} dw_\alpha d\bar{w}_\alpha, \tag{3.24}$$

where $D(m,n)=\frac{1}{2}(m+1)(n+1)(m+n+2)$ is the dimension of the representation. For the choice of shift-up (E_{+i}) or shift-down (E_{-i}) operators[5],

$$E_{\pm 1} := \frac{\lambda_1 \pm i\lambda_2}{2\sqrt{2}}, \quad E_{\pm 2} := \frac{\lambda_4 \pm i\lambda_5}{2\sqrt{2}}, \quad E_{\pm 3} := \frac{\lambda_6 \pm i\lambda_7}{2\sqrt{2}}, \tag{3.25}$$

with the Gell-Mann matrices $\lambda_A (A=1,\cdots,8)$, we obtain

$$\exp\left[\sum_{\alpha=1}^{3}\tau_\alpha E_{-\alpha}\right] = \begin{pmatrix} 1 & w_1 & w_2 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{pmatrix}^t \in F_2 = SU(3)/U(1)^2. \tag{3.26}$$

Two sets of three complex variables are related as (see Appendix A)

$$w_1 = \frac{\tau_1}{\sqrt{2}}, \quad w_2 = \frac{\tau_2}{\sqrt{2}} + \frac{\tau_1\tau_3}{4}, \quad w_3 = \frac{\tau_3}{\sqrt{2}}, \tag{3.27}$$

or conversely

$$\tau_1 = \sqrt{2}w_1, \quad \tau_2 = \sqrt{2}\left(w_2 - \frac{w_1 w_3}{2}\right), \quad \tau_3 = \sqrt{2}w_3. \tag{3.28}$$

The complex projective space CP^2 is covered by three complex plane $\mathbf{C}$ by holomorphic maps [33], see Appendix B. The parameterization of $SU(3)$ in terms of eight angles is possible also in $SU(3)$, just as $SU(2)$ is parameterized by three Euler angles, see [34].

3.3 $SU(N)$ case

For $SU(N)=SU(n+1)$, the flag space [23] is defined by

$$F_n = SU(n+1)/U(1)^n \ni V. \tag{3.29}$$

We use V to denote the element of F_n in this definition. F_n is a compact Kähler manifold [35, 36] which is homogeneous but nonsymmetrical manifold of dimension $dim_{\mathbf{C}} F_n = n(n+1)/2$.

Since the flag manifold F_n is a Kähler manifold [35, 36], it possesses complex *local* coordinates w_α, an Hermitian Riemannian metric,

$$ds^2 = g_{\alpha\bar{\beta}} dw^\alpha d\bar{w}^\beta, \tag{3.30}$$

[5]We take the notation $E_{\pm i}$ for the corresponding root vectors $\pm\alpha^{(i)}$ in Fig.3

and a corresponding two-form, called the Kähler form [6]

$$\Omega_K = i g_{\alpha\bar{\beta}} dw^\alpha \wedge d\bar{w}^\beta, \tag{3.31}$$

which is closed, i.e.,

$$d\Omega_K = 0. \tag{3.32}$$

Any closed form Ω_K is *locally* exact, $\Omega_K = d\omega$ due to Poincaré's lemma. The condition (3.32) is equivalent to

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial w^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial w^\alpha}, \quad \text{or} \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{w}^\gamma} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{w}^\beta}. \tag{3.33}$$

This holds if and only if the metric $g_{\alpha\beta}$ is obtained from a real scalar function K as

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial w^\alpha}\frac{\partial}{\partial \bar{w}^\beta} K, \tag{3.34}$$

where $K = K(w, \bar{w})$ is called the Kähler potential. Then the Kähler two-form is obtained from (3.31),

$$\Omega_K = i\partial\bar{\partial}K. \tag{3.35}$$

On the flag space, there transitively act two groups $G = SU(n+1)$ and also its complexification $G^c = SL(n+1, \mathbf{C})$. Any element of F_n is written as an upper triangular matrix of the size $(n+1)\times(n+1)$, with ones in the main diagonal and off-diagonal $n(n+1)/2$ complex numbers, $w_\alpha \in \mathbf{C}$:

$$\xi = \begin{pmatrix} 1 & w_1 & w_2 & \cdots & \cdots & w_n \\ 0 & 1 & w_{n+1} & \cdots & \cdots & w_{2n-1} \\ 0 & 0 & 1 & w_{2n} & \cdots & w_{3n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & w_{n(n+1)/2} \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in F_n. \tag{3.36}$$

Therefore, we can write

$$F_n = SL(n+1, \mathbf{C})/B_- \ni \xi, \tag{3.37}$$

where $B_-(B_+)$ is the Borel subgroup, i.e., the group of lower (upper) triangular matrices with determinant equal to one (Iwasawa decomposition). This definition (3.37) [7] should be compared with the first definition (3.29). The mapping $G/H \to G^c/B_-$ is a generalization of the stereographic projection in the $G = SU(2)$ case [13]. The action of the group $SL(n+1, \mathbf{C})$ on F_n, $g : V \to V_g$, can be found through the Gauss decomposition:

$$V \cdot g = TDV_g, \quad g \in SL(n+1, \mathbf{C}), \quad T \in Z_-(n+1), \quad V_g \in Z_+(n+1), \tag{3.38}$$

[6] The imaginary unit is needed to make the Kähler two-form real, since $g^-_{\alpha\bar{\beta}} = g_{\bar{\alpha}\beta} = g_{\beta\bar{\alpha}}$.

[7] Note that ξ is not necessarily unitary as a matrix under this definition.

where $Z_+(n+1)(Z_-(n+1))$ is the set of the upper (lower) triangular matrices with units on the main diagonal and D is a diagonal matrix with determinant equal to 1. The elements of the factors T, D, V_g are rational functions of the elements of g. [8]

The group $G = SU(N)$ has the rank $N-1$ and the Cartan subalgebra is constructed from $(N-1)$ diagonal generators H_i. Hence, there are $N(N-1)$ off-diagonal shift operators E_α, since $dimSU(N) := N^2-1 = (N-1)+N(N-1)$. Therefore, the total number of roots is $N(N-1)$, of which there are $N-1$ simple roots, other roots are constructed as linear combination of the simple roots. There are N weight vectors. An element of $SU(N)$ is expressed by the $N \times N$ unitary matrices with unit determinant which is generated by traceless Hermitian matrices, N^2-1 linearly independent generators $T^A (A = 1, \cdots, N^2-1)$. The generators are normalized as

$$\mathrm{tr}(T^A T^B) = \frac{1}{2}\delta_{AB}. \tag{3.40}$$

Each off-diagonal generator E_α has a single non-zero element $1/\sqrt{2}$. The diagonal generator H_m is defined by

$$\begin{aligned} (H_m)_{ab} &= \frac{1}{\sqrt{2m(m+1)}}\left(\sum_{k=1}^{m}\delta_{ak}\delta_{bk} - m\delta_{a,m+1}\delta_{b,m+1}\right) && (3.41)\\ &= \frac{1}{\sqrt{2m(m+1)}}\mathrm{diag}(1, \cdots, 1, -m, 0, \cdots, 0). && (3.42) \end{aligned}$$

For $m = 1$ to $N-1$, H_m has m 1's along the diagonal from upper left-hand corner. The next element is $-m$ to make it traceless. The rest of the diagonal elements (if any) are zero. The weight vector (i.e., eigenvectors of all H_i, $H_j|\nu\rangle = \nu^j|\nu\rangle$) of the fundamental representation $\mathbf{N}$ (N-dimensional irreducible representation of $SU(N)$) is given by [37]

$$\begin{aligned} \nu^1 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2(N-1)N}}\right), \\ \nu^2 &= \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2(N-1)N}}\right), \\ \nu^3 &= \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}, \cdots, \frac{1}{\sqrt{2(N-1)N}}\right), \\ &\vdots \\ \nu^{m+1} &= \left(0, 0, \cdots, 0, -\frac{m}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2(N-1)N}}\right), \end{aligned}$$

[8] For $n = 1$,

$$V = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad V_g = \begin{pmatrix} 1 & \frac{aw+b}{cw+d} \\ 0 & 1 \end{pmatrix}, \tag{3.39}$$

Hence w is the complex one-dimensional representation of $SL(2, C)$.

$$
\vdots
$$

$$
\nu^N = \left(0, 0, \cdots, 0, \frac{-N+1}{\sqrt{2(N-1)N}}\right). \tag{3.43}
$$

All the weight vectors have the same length and the angles between different weights are the same, i.e.,

$$
\nu^i \cdot \nu^i = \frac{N-1}{2N}, \quad \nu^i \cdot \nu^j = -\frac{1}{2N}(i \neq j). \tag{3.44}
$$

The weights constitutes the polygon in the $N-1$ dimensional space. This implies that any weight can be used as the highest weight. A weight will be called positive if its *last* non-zero component is positive. With this definition the weight satisfy

$$
\nu^1 > \nu^2 > \cdots > \nu^N. \tag{3.45}
$$

The simple roots are given by

$$
\alpha^i = \nu^i - \nu^{i+1} \quad (i = 1, \cdots, N-1), \tag{3.46}
$$

namely,

$$
\begin{aligned}
\alpha^1 &= (1, 0, \cdots, 0) \\
\alpha^2 &= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \cdots, 0\right) \\
\alpha^3 &= \left(0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0, \cdots, 0\right) \\
&\vdots \\
\alpha^m &= \left(0, 0, \cdots, -\sqrt{\frac{m-1}{2m}}, \sqrt{\frac{m+1}{2m}}, 0, \cdots, 0\right) \\
&\vdots \\
\alpha^{N-1} &= \left(0, 0, \cdots, -\sqrt{\frac{N-2}{2(N-1)}}, \sqrt{\frac{N}{2(N-1)}}\right).
\end{aligned} \tag{3.47}
$$

As can be shown from (3.44), all these roots have length 1, and angles between successive roots are the same, $2\pi/3$, and other roots are orthogonal, i.e.,

$$
\alpha^j \cdot \alpha^j = 1, \quad \alpha^i \cdot \alpha^j = -\frac{1}{2} \quad (j = i \pm 1), \tag{3.48}
$$

$$
\alpha^i \cdot \alpha^j = 0 \quad (j \neq i, i \pm 1) \tag{3.49}
$$

This fact is usually expressed as the Dynkin diagram, see Fig.5.

If we choose ν^1 as the highest-weight $\vec{\Lambda}$ of the fundamental representation $\mathbf{N}$, some of the roots are orthogonal to ν^1. From the above construction, it is easy to see that

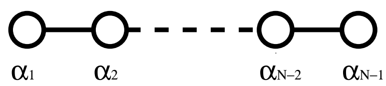

Figure 5: The Dynkin diagram of SU(N).

only one simple root α_1 is non-orthogonal to ν^1, and that all the other simple roots are orthogonal, i.e.,

$$\nu^1 \cdot \alpha^1 \neq 0, \quad \nu^1 \cdot \alpha^2 = \nu^1 \cdot \alpha^3 = \cdots = \nu^1 \cdot \alpha^{N-1} = 0. \tag{3.50}$$

So all the linear combinations constructed from $\alpha^2, \cdots, \alpha^{N-1}$ are also orthogonal to ν^1. Non-orthogonal roots are obtained only when α^1 is included in the linear combinations. It is not difficult to show that the total number of non-orthogonal roots is $2(N-1)$, so there are $N(N-1) - 2(N-1) = (N-2)(N-1)$ orthogonal roots. The $(N-2)(N-1)$ shift operators E_α corresponding to these orthogonal roots together with $N-1$ Cartan subalgebra H_i constitute the maximal stability subgroup $\tilde{H} = U(N-1)$, since $(N-2)(N-1) + (N-1) = (N-1)^2 = dimU(N-1)$. Thus, for the fundamental representation, the stability subgroup $\tilde{H}$ of $SU(N)$ is given by $\tilde{H} = U(N-1)$. In order to describe the coset space $G/\tilde{H}$, we need only $(N-1)$ complex numbers, since

$$G/\tilde{H} = SU(N)/U(N-1), \tag{3.51}$$

and $dimG/\tilde{H} = 2(N-1)$. We conclude that $G/\tilde{H} = CP^{N-1}$, the $(N-1)$-dimensional complex projective space [35, 36, 33, 38] which is a submanifold of the flag manifold F_{N-1}.

The complex projective space CP^n is the compact Kähler symmetric space with $dim_{\mathbf{C}} CP^n = n$. The SU(n+1) group can transitively act on this manifold and this manifold can be considered as a factor space,

$$\mathbf{C}P^n = SU(n+1)/(SU(n) \times U(1)). \tag{3.52}$$

An element of CP^n is expressed using n complex variables $w_1, \cdots, w_n$ as

$$\begin{pmatrix} 1 & w_1 & w_2 & \cdots & \cdots & w_n \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in CP^n. \tag{3.53}$$

The Kähler potential of F_n $(n = N-1)$ is obtained as follows. Let H be the Hermitian matrix defined by $H = VV^\dagger, V \in F_n$. Consider the Gauss decomposition

$$H = VV^\dagger = TDV, \tag{3.54}$$

where $T \in Z_-(n+1)$ and D is a diagonal matrix with determinant equal to 1. H has the upper principal minors $\Delta_\ell(VV^\dagger)(\ell = 1, \cdots, n+1)$ which is equal to the following products of the elements δ_ℓ of the diagonal matrix $D = diag(\delta_1, \cdots, \delta_n)$:

$$\Delta_1 = \delta_1, \quad \Delta_2 = \delta_1\delta_2, \quad \cdots, \quad \Delta_n = \delta_1\delta_2\cdots\delta_n, \quad \Delta_{n+1} = 1. \tag{3.55}$$

Let d_ℓ $(\ell = 1, \cdots, n)$ be the Dynkin index of $SU(n+1)$. Then the Kähler potential of F_n is given in the form,

$$K(w, \bar{w}) = \sum_{\ell=1}^{n} d_\ell K_\ell(w, \bar{w}) = \sum_{\ell=1}^{n} d_\ell \ln \Delta_\ell(w, \bar{w}) = \ln[\prod_{\ell=1}^{n}(\Delta_\ell(w, \bar{w}))^{d_\ell}]. \tag{3.56}$$

The $K(w, \bar{w}')$ is also obtained from the Gauss decomposition of $VV'^\dagger$,

$$K(w, \bar{w}') = \ln[\prod_{\ell=1}^{n}(\Delta_\ell(w, \bar{w}'))^{d_\ell}]. \tag{3.57}$$

The $SU(n+1)$ invariant measure on F_n is written, up to a multiplicative factor, as [39]

$$d\mu(V, \bar{V}) = \rho(V, \bar{V}) \prod_{\alpha=1}^{n(n+1)/2} dw_\alpha d\bar{w}_\alpha, \tag{3.58}$$

$$\rho(V, \bar{V}) = [\prod_{\ell=1}^{n} \Delta_\ell^{d_\ell}]^{-2} = [\prod_{\ell=1}^{n} \Delta_\ell]^{-2}, \tag{3.59}$$

where all $d_\ell = 1$. The density ρ of the invariant measure is calculated [23] from

$$\rho = \det(g_{\alpha\bar{\beta}}) = \det\left(\frac{\partial^2 K}{\partial w^\alpha \partial \bar{w}^\beta}\right). \tag{3.60}$$

The Kähler potential of CP^{N-1} manifold is given by

$$K = m \ln\left(1 + \sum_{\alpha=1}^{N-1} |w_\alpha|^2\right) = m \ln\left(1 + |||w|||^2\right), \tag{3.61}$$

where

$$|||w|||^2 := \sum_{a=1}^{N-1} |w_a|^2. \tag{3.62}$$

Hence, the metric reads

$$g_{\alpha\bar{\beta}} = m\frac{(1+|||w|||^2)\delta_{\alpha\beta} - \bar{w}_\alpha w_\beta}{(1+|||w|||^2)^2}. \tag{3.63}$$

The above construction of the coherent state can be extended into arbitrary compact semi-simple Lie group, see Chapter 11 of Perelomov [40].

4 Non-Abelian Stokes theorem

In this section we derive a new version of non-Abelian Stokes theorem (NAST) based on the coherent state obtained in the previous section. An advantage of this version of NAST is that it is possible to separate the magnetic monopole contribution in the Wilson loop and that it is very helpful to understand the dual superconductor picture of quark confinement in QCD.

4.1 Non-Abelian Stokes theorem for $SU(N)$

We consider the infinitesimal deviation $\xi' = \xi + d\xi$ (which is sufficient to derive the path integral formula). From (3.19), i.e.,

$$|\xi,\Lambda\rangle = \xi|\Lambda\rangle = e^{-\frac{1}{2}K(w,\bar{w})} \exp\left[\sum_\alpha \tau_\alpha(w) E_{-\alpha}\right] |\Lambda\rangle, \tag{4.1}$$

we find

$$\begin{aligned} \langle \xi + d\xi, \Lambda | \xi, \Lambda \rangle &= \exp[i\omega + O((dw)^2)], & (4.2) \\ \omega(x) &:= \langle \Lambda | i\xi^\dagger(x) d\xi(x) | \Lambda \rangle, & (4.3) \end{aligned}$$

where d denotes an exterior derivative,

$$d := dx^\mu \frac{\partial}{\partial x^\mu} := dx^\mu \partial_\mu. \tag{4.4}$$

Then ω is the one-form,

$$\omega = dx^\mu \omega_\mu, \quad \omega_\mu = \langle \Lambda | i\xi^\dagger(x) \partial_\mu \xi(x) | \Lambda \rangle. \tag{4.5}$$

Here the x-dependence of ξ comes through that of $w(x)$ (the local field variable $w(x)$), i.e., $\xi(x) = \xi(w(x))$.

The exterior derivative is also regarded as the operator,

$$d = \partial + \bar{\partial} = dw_\alpha \frac{\partial}{\partial w_\alpha} + d\bar{w}_\beta \frac{\partial}{\partial \bar{w}_\beta}, \tag{4.6}$$

where the operators ∂ and $\bar{\partial}$ are called the Dolbeault operators [35]. From the inner product (3.22),

$$\langle \xi', \Lambda | \xi, \Lambda \rangle = e^{K(w,\bar{w}')} e^{-\frac{1}{2}[K(w',\bar{w}')+K(w,\bar{w})]}, \tag{4.7}$$

we obtain another expression for ω using the Kähler potential K,

$$\omega = \frac{i}{2}(\partial - \bar{\partial})K = \frac{i}{2}\left(dw_\alpha \frac{\partial}{\partial w_\alpha} - d\bar{w}_\beta \frac{\partial}{\partial \bar{w}_\beta}\right) K. \tag{4.8}$$

The Wilson loop operator $W^C[\mathcal{A}]$ is defined as the trace of the path-ordered exponent along the closed loop C,

$$W^C[\mathcal{A}] := \frac{1}{\mathcal{N}} \mathrm{tr}\left[\mathcal{P}\exp\left(ig \oint_C \mathcal{A}\right)\right], \tag{4.9}$$

where $\mathcal{N}$ is the dimension of the representation, i.e., $\mathcal{N} = \dim(\mathbf{1}_R) = \mathrm{tr}(\mathbf{1}_R)$, and $\mathcal{A}$ is the (Lie-algebra valued) connection one-form,

$$\mathcal{A}(x) = \mathcal{A}^A_\mu(x) T^A dx^\mu = \mathcal{A}^A(x) T^A. \tag{4.10}$$

Consider a curve starting from x_0 and ending at x. We parameterize the curve by the parameter t and define

$$W_{ab}(t, t_0) := \left[\mathcal{P}\exp\left(ig \int_{x_0(t_0)}^{x(t)} dx^\mu \mathcal{A}_\mu(x)\right)\right]_{ab} = \left[\mathcal{P}\exp\left(ig \int_{t_0}^{t} dt \mathcal{A}(t)\right)\right]_{ab}, \tag{4.11}$$

where

$$\mathcal{A}(t) := \mathcal{A}_\mu(x) dx^\mu / dt. \tag{4.12}$$

Then the wavefunction defined by

$$\psi_a(t) = W_{ab}(t, t_0)\psi_b(t_0) \tag{4.13}$$

is a solution of the Schrödinger equation,

$$\left[i\frac{d}{dt} + g\mathcal{A}(t)\right]_{ab} \psi_b(t) = 0. \tag{4.14}$$

Note that the Wilson loop operator is obtained by taking the trace of (4.11) for the closed loop, say C,

$$W^C[\mathcal{A}] = \frac{1}{\mathcal{N}} \mathrm{tr}(W(t, 0)) = \frac{1}{\mathcal{N}} \sum_{a=1}^{N} W_{aa}(t, 0). \tag{4.15}$$

This implies [21] that it is possible to write the path integral representation of the Wilson loop operator if we identify $\mathcal{A}(t)$ with the Hamiltonian,

$$H(t) = g\mathcal{A}(t) = g\mathcal{A}_\mu(x) dx^\mu / dt. \tag{4.16}$$

Following [15], the path-ordered exponent is defined by discretizing the time interval t into N infinitesimal steps and subsequently taking the limit $N \to \infty, \epsilon \to 0$ with $N\epsilon = t$ being kept fixed,

$$\mathrm{tr}\left\{\mathcal{P}\exp\left[ig\int_0^t dt\mathcal{A}(t)\right]\right\} = \lim_{N\to\infty,\epsilon\to 0}\mathrm{tr}\prod_{n=0}^{N-1}[1+i\epsilon g\mathcal{A}(t_n)], \tag{4.17}$$

where $t_n := n\epsilon, \epsilon := t/N$. For simplicity, we take $t_0 = 0$ and $t_N = t$. In the R.H.S. of (4.17), we replace the trace with

$$\frac{1}{\mathcal{N}}\mathrm{tr}(\cdots) = \int d\mu(\xi_N)\langle\xi_N,\Lambda|(\cdots)|\xi_N,\Lambda\rangle, \tag{4.18}$$

and insert the complete set (resolution of unity),

$$I = \int |\xi_n,\Lambda\rangle d\mu(\xi_n)\langle\xi_n,\Lambda| \quad (n = 1,2,\cdots,N-1). \tag{4.19}$$

Then we obtain

$$\begin{aligned}
& \frac{1}{\mathcal{N}}\mathrm{tr}\left\{\mathcal{P}\exp\left[ig\int_0^t dt\mathcal{A}(t)\right]\right\} \\
= & \lim_{N\to\infty,\epsilon\to 0}\int\cdots\int d\mu(\xi_N)\langle\xi_N,\Lambda|[1+i\epsilon g\mathcal{A}(t_{N-1})]|\xi_{N-1},\Lambda\rangle d\mu(\xi_{N-1}) \\
& \times\langle\xi_{N-1},\Lambda|[1+i\epsilon g\mathcal{A}(t_{N-2})]|\xi_{N-2},\Lambda\rangle d\mu(\xi_{N-2}) \\
& \cdots d\mu(\xi_1)\langle\xi_1,\Lambda|[1+i\epsilon g\mathcal{A}(t_0)]|\xi_N,\Lambda\rangle \\
= & \lim_{N\to\infty,\epsilon\to 0}\prod_{n=1}^{N}\int d\mu(\xi_n)\prod_{n=0}^{N-1}\langle\xi_{n+1},\Lambda|[1+i\epsilon g\mathcal{A}(t_n)]|\xi_n,\Lambda\rangle \\
= & \lim_{N\to\infty,\epsilon\to 0}\prod_{n=1}^{N}\int d\mu(\xi(t_n))\prod_{n=0}^{N-1}\left[1+i\epsilon g\bar{A}(t_n)\right]\prod_{n=0}^{N-1}\langle\xi(t_{n+1}),\Lambda|\xi(t_n),\Lambda\rangle \\
= & \lim_{N\to\infty,\epsilon\to 0}\prod_{n=1}^{N}\int d\mu(\xi(t_n))\exp\left[i\epsilon\sum_{n=0}^{N-1}g\bar{A}(t_n)\right]\prod_{n=0}^{N-1}\langle\xi(t_{n+1}),\Lambda|\xi(t_n),\Lambda\rangle,
\end{aligned} \tag{4.20}$$

where we have used $\xi_0 = \xi_N$ and have defined

$$\bar{A}(t_n) := \frac{\langle\xi_{n+1},\Lambda|\mathcal{A}(t_n)|\xi_n,\Lambda\rangle}{\langle\xi_{n+1},\Lambda|\xi_n,\Lambda\rangle}. \tag{4.21}$$

Up to $O(\epsilon^2)$, we find

$$\bar{A}(t_n) := \langle\xi_n,\Lambda|\mathcal{A}(t_n)|\xi_n,\Lambda\rangle + O(\epsilon^2) = \langle\Lambda|\xi(t_n)^\dagger\mathcal{A}(t_n)\xi(t_n)|\Lambda\rangle + O(\epsilon^2), \tag{4.22}$$

and

$$\begin{aligned}
\langle\xi_{n+1},\Lambda|\xi_n,\Lambda\rangle &= \langle\xi(t_n),\Lambda|\xi(t_n),\Lambda\rangle + \epsilon\langle\dot{\xi}(t_n),\Lambda|\xi(t_n),\Lambda\rangle + O(\epsilon^2) \\
&= \exp[\epsilon\langle\dot{\xi}(t_n),\Lambda|\xi(t_n),\Lambda\rangle + O(\epsilon^2)] \\
&= \exp[-i\epsilon\langle\Lambda|i\dot{\xi}(t_n)^\dagger\xi(t_n)|\Lambda\rangle + O(\epsilon^2)] \\
&= \exp[i\epsilon\langle\Lambda|i\xi(t_n)^\dagger\dot{\xi}(t_n)|\Lambda\rangle + O(\epsilon^2)],
\end{aligned} \tag{4.23}$$

where we have used $\langle \xi(t_n), \Lambda | \xi(t_n), \Lambda \rangle = 1$. Therefore we arrive at the expression,

$$\begin{aligned} W^C[\mathcal{A}] &= \lim_{N\to\infty,\epsilon\to 0} \prod_{n=1}^{N} \int d\mu(\xi(t_n)) \\ &\quad \times \exp\left\{ ig\epsilon \sum_{n=0}^{N-1} \langle \Lambda | [\xi(t_n)^\dagger \mathcal{A}(t_n)\xi(t_n) + ig^{-1}\xi(t_n)^\dagger \dot{\xi}(t_n)] | \Lambda \rangle \right\}. \end{aligned} \quad (4.24)$$

Thus we obtain the path integral representation of the Wilson loop,

$$W^C[\mathcal{A}] = \int [d\mu(\xi)]_C \exp\left(ig \oint_C \langle \Lambda | \left[V\mathcal{A}V^\dagger + \frac{i}{g} V dV^\dagger \right] | \Lambda \rangle \right), \quad (4.25)$$

where $[d\mu(\xi)]_C$ is the product measure of $d\mu(w(x), \bar{w}(x))$ along the loop C,

$$[d\mu(\xi)]_C := \lim_{N\to\infty,\epsilon\to 0} \prod_{n=1}^{N} \int d\mu(\xi_n). \quad (4.26)$$

Using the (usual) Stokes theorem $\oint_{C=\partial S} \omega = \int_S d\omega$, we obtain the non-Abelian Stokes theorem (NAST):

$$\begin{aligned} W^C[\mathcal{A}] &= \int [d\mu(\xi)]_C \exp\left(ig \oint_C \left[n^A \mathcal{A}^A + \frac{1}{g}\omega \right] \right) \\ &= \int [d\mu(\xi)]_C \exp\left(ig \int_{S:\partial S = C} \left[d(n^A \mathcal{A}^A) + \frac{1}{g}\Omega_K \right] \right), \end{aligned} \quad (4.27)$$

where we have defined

$$n^A(x) := \langle \Lambda | \xi^\dagger(x) T^A \xi(x) | \Lambda \rangle, \quad (4.28)$$

$$\omega(x) := \langle \Lambda | i\xi^\dagger(x) d\xi(x) | \Lambda \rangle, \quad (4.29)$$

and

$$\Omega_K := d\omega. \quad (4.30)$$

Taking into account (4.8), we find that this Ω_K is nothing but the Kähler two-form in agreement with the general statement (3.35), i.e.,

$$\Omega_K = d\omega = (\partial + \bar{\partial})\frac{i}{2}(\partial - \bar{\partial})K = i\partial\bar{\partial}K, \quad (4.31)$$

since the identity $d^2 = 0$ leads to $\partial^2 = 0 = \bar{\partial}^2, \partial\bar{\partial} + \bar{\partial}\partial = 0$. Therefore, the second term ω or Ω_K in the exponent in the NAST is entirely determined from the Kähler potential of the flag manifold.

For $SU(N)$, the topological part,

$$\gamma := \oint_C \omega = \int_S \Omega_K, \quad (4.32)$$

corresponding to the residual $U(N-1)$ invariance is interpreted as the geometric phase of the Wilczek-Zee holonomy [41], just as it is interpreted in the SU(2) case as the Berry-Aharonov-Anandan phase for the residual U(1) invariance. The details will be given in a subsequent article [42].

4.2 Fundamental representation of $SU(N)$ and CP^{N-1} variable

Defining

$$\omega_a(x) := (V^\dagger(x)|\Lambda\rangle)_a \quad (a = 1, \cdots, N), \tag{4.33}$$

we can write

$$n^A(x) := \langle\Lambda|V(x)T^A V^\dagger(x)|\Lambda\rangle = \bar{\omega}_a(x)(T^A)_{ab}\omega_b(x). \tag{4.34}$$

In the CP^{N-1} case, especially, the highest-weight state is given by a column vector,

$$|\Lambda\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.35}$$

and then we can write

$$n^A(x) := \langle\Lambda|U(x)T^A U^\dagger(x)|\Lambda\rangle = \bar{\phi}_a(x)(T^A)_{ab}\phi_b(x), \tag{4.36}$$

where $U \in SU(N)$ and

$$\phi_a(x) \quad := \quad (U^\dagger(x)|\Lambda\rangle)_a = \bar{U}_{1a}(x). \tag{4.37}$$

Then $n^A(x)$ can be rewritten into

$$n^A(x) \quad = \quad (U(x)T^A U^\dagger(x))_{11}. \tag{4.38}$$

Note that the CP^{N-1} variables $\phi_a(a = 1, \cdots, N)$ are subject to the constraint,

$$\sum_{a=1}^{N} |\phi_a(x)|^2 = 1. \tag{4.39}$$

This is clearly satisfied by the unitarity of U, $\sum_{a=1}^{N} U_{1a}(x)\bar{U}_{1a}(x) = (U(x)U^\dagger(x))_{11} = 1$.

On the other hand, we examine another expression (adjoint orbit representation),

$$n^A(x) = 2\mathrm{tr}(U^\dagger(x)\mathcal{H}U(x)T^A), \tag{4.40}$$

or equivalently,

$$\mathbf{n}(x) := n^A(x)T^A = U^\dagger(x)\mathcal{H}U(x), \tag{4.41}$$

where $\mathcal{H}$ is defined by

$$\mathcal{H} = \vec{\Lambda} \cdot (H^1, \cdots, H^{N-1}) = \sum_{i=1}^{N-1} \Lambda^i H^i = \frac{1}{2}\mathrm{diag}\left(\frac{N-1}{N}, -\frac{1}{N}, \cdots, -\frac{1}{N}\right), \tag{4.42}$$

where we have used (3.42) and

$$\Lambda^i = \frac{1}{\sqrt{2i(i+1)}}. \tag{4.43}$$

For $SU(3)$, when the Dynkin index $[m,n]=[1,0]$ or $[0,1]$, (4.42) reduces to

$$\mathcal{H}=\vec{\Lambda}\cdot(\frac{\lambda^3}{2},\frac{\lambda^8}{2})=\frac{1}{2}\mathrm{diag}(\frac{2}{3},-\frac{1}{3},-\frac{1}{3}),\quad \text{or}\quad \frac{1}{2}\mathrm{diag}(\frac{1}{3},\frac{1}{3},-\frac{2}{3}). \tag{4.44}$$

Note that two elements agree with each other. Hence the adjoint orbit can not cover all the flag space F_2. This is the CP^2 case. It is easy to see that two definitions (4.36) and (4.40) are equivalent,

$$n^A(x)=(U(x)T^AU^\dagger(x))_{11}=2\mathrm{tr}(\mathcal{H}U(x)T^AU^\dagger(x)), \tag{4.45}$$

since $UT^AU^\dagger$ is traceless. [9]

A correspondence between the F_{N-1} variables w_a and the CP^{N-1} variables ϕ_a is given, e.g., by

$$\phi_1=w_1,\phi_2=w_2,\cdots,\phi_{N-1}=w_{N-1},\phi_N=1, \tag{4.46}$$

i.e., ω is the inhomogeneous coordinate, $\omega_a=\phi_a/\phi_N=w_a$ $(a=1,\cdots,N-1)$ by definition. In the CP^{N-1} case (for the fundamental representation), we can perform the following replacement,

$$\langle\Lambda|f(V)|\Lambda\rangle=2\mathrm{tr}[\mathcal{H}f(U)], \tag{4.47}$$

if $f(V)$ is traceless, i.e., $\mathrm{tr}f(V)=0$. Hence, we obtain another expression for ω,

$$\omega(x):=2\mathrm{tr}[\mathcal{H}iU(x)dU^\dagger(x)]=-i2\mathrm{tr}[\mathcal{H}dU(x)U^\dagger(x)], \tag{4.48}$$

which is a diagonal piece of the Maurer-Cartan one-form,

$$\vartheta:=dUU^{-1}. \tag{4.49}$$

It turns out that the two-form Ω_K is the symplectic two-form [43],

$$\Omega_K=d\omega=2\mathrm{tr}(\mathcal{H}[U^{-1}dU,U^{-1}dU])=2\mathrm{tr}(\mathbf{n}[d\mathbf{n},d\mathbf{n}]). \tag{4.50}$$

Our choice of $\mathcal{H}$ is the most economical one, see [43] for different choices and more arguments on the related issues.

From the Kähler potential of CP^{N-1} (3.61) and the relation (4.8), the connection one-form ω reads

$$\omega:=\frac{i}{2}\frac{\bar{w}_\alpha dw_\alpha-d\bar{w}_\alpha w_\alpha}{1+\bar{w}_\alpha w_\alpha}, \tag{4.51}$$

which is equal to

$$\omega:=i\frac{\bar{w}_\alpha dw_\alpha}{1+\bar{w}_\alpha w_\alpha}, \tag{4.52}$$

up to the total derivative. By taking the exterior derivative, we obtain

$$\Omega_K=d\omega=i\frac{(1+|||w|||^2)\delta_{\alpha\beta}-\bar{w}_\alpha w_\beta}{(1+|||w|||^2)^2}dw^\alpha\wedge d\bar{w}^\beta, \tag{4.53}$$

which agrees with the metric (3.63).

[9] When $[m,n]=[1,1]$, on the other hand, $\mathcal{H}=\mathrm{diag}(1,-1,0)$, and all the diagonal elements are different. Therefore $\mathbf{n}(x)$ moves on the whole flag space, F_2.

4.3 An implication of the NAST

The NAST (4.27) implies that the Wilson loop operator is given by

$$W^C[\mathcal{A}] = \int [d\mu(\xi)]_C \exp\left(ig\oint_C a\right) = \int [d\mu(\xi)]_C \exp\left(ig\int_{S:C=\partial S} f\right). \qquad (4.54)$$

First, a is the connection one-form,

$$a := n^A \mathcal{A}^A + \frac{1}{g}\omega = \langle \Lambda | \mathcal{A}^V | \Lambda \rangle, \qquad (4.55)$$

where $\mathcal{A}^V$ is the gauge transformation of $\mathcal{A}$ by $V \in F_{N-1}$,

$$\mathcal{A}^V := V\mathcal{A}V^\dagger + \frac{i}{g}VdV^\dagger = \xi^\dagger \mathcal{A}\xi + \frac{i}{g}\xi^\dagger d\xi. \qquad (4.56)$$

For quark in the fundamental representation, we can write

$$a = 2\mathrm{tr}(\mathcal{H}\mathcal{A}^V). \qquad (4.57)$$

Therefore, the one-form a is equal to the diagonal piece of the gauge-transformed potential $\mathcal{A}^V$. This fact is very useful to derive the Abelian dominance in the low-energy physics of QCD, see section 6.

Next, f is the curvature two-form,

$$f := da = dC + \frac{1}{g}d\omega = dC + \frac{1}{g}\Omega_K, \qquad (4.58)$$

where we defined the one-form,

$$C := n^A \mathcal{A}^A. \qquad (4.59)$$

The anti-symmetric tensor $f_{\mu\nu}$ can be called the generalized 't Hooft-Polyakov tensor by the following reasons; (1) it gives non-vanishing magnetic monopole (current) where only the second term Ω_K gives a non-trivial contribution. (2) it is invariant under the full gauge transformation, although it is an Abelian field strength. These facts are shown as follows.

First of all, we characterize the flag space in the complex coordinates, namely, the target space at each space-time point $x \in R^D$ is parameterized by the complex variables, $w^\alpha = w^\alpha(x)$. The Kähler two-form is rewritten as

$$\Omega_K = ig_{\alpha\bar{\beta}}\partial_\mu w^\alpha \partial_\nu \bar{w}^\beta dx^\mu \wedge dx^\nu. \qquad (4.60)$$

On the other hand,

$$\Omega_K := d\omega = \frac{1}{2}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)dx^\mu \wedge dx^\nu := \frac{g}{2}f^\Omega_{\mu\nu}dx^\mu \wedge dx^\nu. \qquad (4.61)$$

Then the second piece $g^{-1}\Omega_K$ of f is written as

$$f^\Omega_{\mu\nu} = \frac{1}{g}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu) = \frac{i}{g}g_{\alpha\bar{\beta}}\partial_\mu w^\alpha \partial_\nu \bar{w}^\beta, \qquad (4.62)$$

and hence

$$f_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + f^\Omega_{\mu\nu}. \tag{4.63}$$

The magnetic monopole current k_μ is obtained as the divergence of the dual tensor ${}^*f^\Omega_{\mu\nu}$,

$$k_\mu := \partial_\nu {}^*f_{\mu\nu}, \tag{4.64}$$

where the Hodge dual of $f_{\mu\nu}$ in four dimensions is defined by

$${}^*f_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}. \tag{4.65}$$

The first piece dC in f does not contribute to the magnetic current, due to the Bianchi identity. On the other hand, the second term Ω_K in f can lead to the non-vanishing magnetic current, as shown shortly. Here it should be remarked that the expression for Ω_K given in terms of the *local* coordinate w_α leads to vanishing magnetic current. In fact, if the metric is given by the Kähler potential,

$$\begin{aligned}
k_\mu &= \frac{i}{2g}\epsilon_{\mu\nu\rho\sigma}\partial_\nu(g_{\alpha\bar\beta}\partial_\rho w^\alpha \partial_\sigma \bar w^\beta) \\
&= \frac{i}{2g}\epsilon_{\mu\nu\rho\sigma}\partial_\nu g_{\alpha\bar\beta}\partial_\rho w^\alpha \partial_\sigma \bar w^\beta \\
&= \frac{i}{2g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{\partial g_{\alpha\bar\beta}}{\partial w^\gamma}\partial_\nu w^\gamma + \frac{\partial g_{\alpha\bar\beta}}{\partial \bar w^\gamma}\partial_\nu \bar w^\gamma\right)\partial_\rho w^\alpha \partial_\sigma \bar w^\beta, \\
&= \frac{i}{2g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{\partial K}{\partial w^\gamma \partial w^\alpha \partial \bar w^\beta} + \frac{\partial K}{\partial \bar w^\gamma \partial w^\alpha \partial \bar w^\beta}\right)\partial_\nu \bar w^\gamma \partial_\rho w^\alpha \partial_\sigma \bar w^\beta = 0,
\end{aligned} \tag{4.66}$$

where we have used the antisymmetric property of $\epsilon_{\mu\nu\rho\sigma}$ under the exchange of ν and ρ, and ν and σ. However, this does not imply the vanishing total magnetic flux or magnetic charge.

We recall that this situation is similar to that of Wu-Yang monopole [45] compared with the original Dirac monopole [44]. There are two ways for describing the Dirac magnetic monopole. One is to use a single vector potential with (line) singularities called the Dirac string where the singularities are distributed on a semi-infinite line extending from the origin of the space coordinates. In the absence of singularities, the vector potential gives vanishing magnetic charge, due to Bianchi identity,

$$\Phi = \oint_{S^2} \mathbf{B}\cdot d\mathbf{S} = \oint_{S^2} \mathrm{curl}\mathbf{A}\cdot d\mathbf{S} = \int_{D^3} \mathrm{divcurl}\mathbf{A} dV = 0. \tag{4.67}$$

Therefore, the singularities must produce the magnetic charge which has the same magnitude as the magnetic charge at the origin, but the opposite sign. Another way is to introduce more than one vector potential to avoid singularities. Each vector potential A^α is defined in a subregion U_α of the sphere S^2 such that A^α is regular in each region U_α and the union of the subregions covers the whole sphere, i.e., $S^2 \subset \cup_\alpha U_\alpha$. Therefore, the Bianchi identity leads to zero magnetic flux in each subregion. Note that we can not apply the Gauss theorem $\mathrm{divcurl}\mathbf{A} = 0$, since U_α is not a closed surface. The total magnetic flux is recovered by summing up all the contributions of the difference of two vector potentials on the boundary $B_{\alpha,\beta}$ shared by two regions, U_α and U_β,

$$\Phi = \sum_\alpha \oint_{U_\alpha} \mathrm{curl}\mathbf{A}^\alpha \cdot d\mathbf{S} = \sum_{\alpha,\beta} \oint_{B_{\alpha,\beta}} (\mathbf{A}^\alpha - \mathbf{A}^\beta) d\ell, \tag{4.68}$$

where the minus sign follows from the fact that the orientation of the boundary is opposite for the neighboring regions. The difference is given by the gauge transformation, $\mathbf{A}^\alpha(x) - \mathbf{A}^\beta(x) = \nabla\Lambda_{\alpha,\beta}(x)$. This recovers the same magnetic flux as the former case.

In view of this, the variable w^α corresponds to $\mathbf{A}^\alpha$ in the Wu-Yang monopole. Therefore, to show the existence of non-zero magnetic flux, we must specify how to glue different coordinates patches on the boundary. These subtleties are avoided by using different parameterization. This generalizes the argument given by 't Hooft and Polyakov for the $SU(2)$ magnetic monopole. The antisymmetric tensor $f_{\mu\nu}$ given by (4.58) is the $SU(N)$ generalization of the 't Hooft-Polyakov tensor for $SU(2)$. In the $SU(2)$ case, $a = 2\mathrm{tr}(T^3\mathcal{A}^V)$ for any representation and the two-form $f := da$ is the Abelian field strength which is invariant under the SU(2) transformation. Hence the two-form f is nothing but the 't Hooft-Polyakov tensor

$$f_{\mu\nu}(x) := \partial_\mu(n^A(x)\mathcal{A}_\nu^A(x)) - \partial_\nu(n^A(x)\mathcal{A}_\mu^A(x)) - \frac{1}{g}\mathbf{n}(x)\cdot(\partial_\mu\mathbf{n}(x)\times\partial_\nu\mathbf{n}(x)), \tag{4.69}$$

describing the magnetic flux emanating from the magnetic monopole, if we identify n^A with the direction of the Higgs field,

$$\hat{\phi}^A := \phi^A/|\phi|, \quad |\phi| := \sqrt{\phi^A\phi^A}. \tag{4.70}$$

The complex coordinate representation reads

$$f_{\mu\nu}^\Omega(x) = \frac{1}{g}\frac{1}{(1+|w(x)|^2)^2}\partial_\mu w(x)\partial_\nu \bar{w}(x). \tag{4.71}$$

In general, the (curvature) two-form $f = d(n^A\mathcal{A}^A) + \Omega_K$ in the NAST is the Abelian field strength which is invariant under the full non-Abelian gauge transformation of $G = SU(N)$, [10]

$$f_{\mu\nu}(x) := \partial_\mu(n^A(x)\mathcal{A}_\nu^A(x)) - \partial_\nu(n^A(x)\mathcal{A}_\mu^A(x)) + \frac{i}{g}\mathbf{n}(x)\cdot[\partial_\mu\mathbf{n}(x),\partial_\nu\mathbf{n}(x)]. \tag{4.75}$$

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[10] The normalization,

$$\mathrm{tr}(T^AT^B) = \frac{1}{2}\delta_{AB}, \tag{4.72}$$

holds for any group. For SU(2), $\mathrm{tr}(T^AT^BT^C) = \frac{1}{4}i\epsilon_{ABC}$. For SU(3), $\mathrm{tr}(T^A[T^B,T^C]) = \frac{1}{4}if_{ABC}$, while

$$\mathrm{tr}(T^AT^BT^C) = \frac{1}{4}(if_{ABC} + d_{ABC}). \tag{4.73}$$

Here we have used $T^BT^C = \frac{1}{2}[T^B,T^C] + \frac{1}{2}\{T^B,T^C\}$, $[T^B,T^C] = if_{BCD}T^D$, and $\{T^B,T^C\} = \frac{1}{3}\delta_{AB}I + d_{BCD}T^D$, where d_{ABC} is completely symmetric in the indices. Furthermore, we find

$$\mathrm{tr}(T^AT^BT^CT^D) = \frac{1}{12}\delta_{AB}\delta_{CD} - \frac{1}{8}f_{ABE}f_{CDE} + \frac{1}{8}d_{ABE}d_{CDE} + \frac{i}{8}(f_{ABE}d_{CDE} + f_{CDE}d_{ABE}). \tag{4.74}$$

The invariance of f is obvious from the NAST (4.54), since $W^C[\mathcal{A}]$ is gauge invariant and the measure $[d\mu(\xi)]_C$ is also invariant under the G gauge transformation. In the case of fundamental representation, the invariance is easily seen, because it is possible to rewrite (4.58) or (4.75) into the manifestly gauge invariant form [46]:

$$f_{\mu\nu}(x) := 2\mathrm{tr}\left(\mathbf{n}(x)\mathcal{F}_{\mu\nu}(x) + \frac{i}{g}\mathbf{n}(x)[D_\mu\mathbf{n}(x), D_\nu\mathbf{n}(x)]\right), \tag{4.76}$$

where

$$\mathcal{F}_{\mu\nu}(x) := \partial_\mu\mathcal{A}_\nu(x) - \partial_\nu\mathcal{A}_\mu(x) - ig[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)], \tag{4.77}$$

and

$$D_\mu\mathbf{n}(x) := \partial_\mu\mathbf{n}(x) - ig[\mathcal{A}_\mu(x), \mathbf{n}(x)]. \tag{4.78}$$

In fact, we obtain the magnetic charge,

$$\begin{aligned} g_m &= \int_{D^3} d^3x k_0 & (4.79)\\ &= \int_{D^3} d^3x \frac{1}{2}\epsilon_{ijk}\partial_i f_{jk} \quad (i,j,k=1,2,3) \\ &= \int_{D^3} d^3x \frac{1}{2}\epsilon_{ijk}\partial_i(\frac{i}{g}\mathbf{n}\cdot[\partial_j\mathbf{n}, \partial_k\mathbf{n}]) \\ &= \int d^2\sigma_i \frac{1}{2}\epsilon_{ijk}\frac{i}{g}\mathbf{n}\cdot[\partial_j\mathbf{n}, \partial_k\mathbf{n}] \\ &= \int_{S^2} d^2x \frac{1}{2}\epsilon_{ab}\frac{i}{g}\mathbf{n}\cdot[\partial_a\mathbf{n}, \partial_b\mathbf{n}] \quad (a,b=1,2) \\ &= \frac{1}{g}\int \Omega_K =: \frac{1}{g}\pi Q, & (4.80) \end{aligned}$$

where we have used (4.50) in the last step. Here Q is the integer-valued instanton charge in the NLS model in two-dimensional space $S^2 = \mathbf{R}^2 \cup \{\infty\}$, see section 8. The contribution from the magnetic monopole is replaced by the instanton in two-dimensional NLS model. The magnetic charge satisfies the Dirac quantization condition,

$$g_m g = \pi Q = \pi n \quad (n = 0, \mp 1, \mp 2, \cdots). \tag{4.81}$$

4.4 Explicit forms of ω and Ω_K for $SU(3)$ and $SU(2)$

For SU(3), we find that ω is given by

$$\omega = im\frac{w_1 d\bar{w}_1 + w_2 d\bar{w}_2}{\Delta_1(w,\bar{w})} + in\frac{w_3 d\bar{w}_3 + (w_2 - w_1 w_3)(d\bar{w}_2 - \bar{w}_1 d\bar{w}_3 - \bar{w}_3 d\bar{w}_1)}{\Delta_2(w,\bar{w})}, \tag{4.82}$$

up to the total derivative. Hence, we obtain

$$\begin{aligned} \Omega_K = d\omega &= im(\Delta_1)^{-2}[(1+|w_1|^2)dw_2\wedge d\bar{w}_2 - \bar{w}_2 w_1 dw_2 \wedge d\bar{w}_1 \\ &-w_2\bar{w}_1 dw_1\wedge d\bar{w}_2 + (1+|w_2|^2)dw_1\wedge d\bar{w}_1] \\ &+in(\Delta_2)^{-2}[\Delta_1 dw_3\wedge d\bar{w}_3 - (w_1+\bar{w}_3 w_2)dw_3\wedge(d\bar{w}_2 - \bar{w}_3 d\bar{w}_1) \\ &-(\bar{w}_1 + w_3\bar{w}_2)(dw_2 - w_3 dw_1)\wedge d\bar{w}_3 \\ &+(1+|w_3|^2)(dw_2 - w_3 dw_1)(d\bar{w}_2 - \bar{w}_3 d\bar{w}_1)]. \end{aligned} \tag{4.83}$$

The Kähler potential for F_2 is given by

$$K(w,\bar{w}) = \ln[(\Delta_1)^m(\Delta_2)^n]. \tag{4.84}$$

For CP^2, it reads

$$K(w,\bar{w}) = \ln[(\Delta_1)^m] \tag{4.85}$$

which is obtained as a special case of F_2 by putting $w_3 = 0$ and $n = 0$. Hence, we obtain

$$\omega = im\frac{w_1 d\bar{w}_1 + w_2 d\bar{w}_2}{\Delta_1(w,\bar{w})}, \tag{4.86}$$

up to the total derivative, and

$$\begin{aligned}\Omega_K &= d\omega = im(\Delta_1)^{-2}[(1+|w_1|^2)dw_2 \wedge d\bar{w}_2 - \bar{w}_2 w_1 dw_2 \wedge d\bar{w}_1 \\ &\quad -w_2\bar{w}_1 dw_1 \wedge d\bar{w}_2 + (1+|w_2|^2)dw_1 \wedge d\bar{w}_1]. \end{aligned} \tag{4.87}$$

This should be compared with the case of $F_1 = CP^1$,

$$K(w,\bar{w}) = m\ln[(1+|w|^2)], \quad m = 2j. \tag{4.88}$$

For SU(2), we reproduce the well-known results;

$$\omega = im\frac{wd\bar{w}}{1+|w|^2}, \tag{4.89}$$

and

$$\Omega_K = ig_{w\bar{w}} dw \wedge d\bar{w} = im\frac{dw \wedge d\bar{w}}{(1+|w|^2)^2}. \tag{4.90}$$

The explicit form of Ω_K is necessary to perform the instanton calculus in the following.

5 Magnetic monopole in SU(N) Yang-Mills theory

In the dual superconductor picture of quark confinement, the magnetic monopoles give the dominant contribution to the area law of the Wilson loop or the string tension. Following the 't Hooft argument [10], the partial gauge fixing $G \to H$ realizes the magnetic monopole in Yang-Mills gauge theory even in the absence of elementary scalar field. In the conventional approach based on the MA gauge, the residual gauge group was chosen to be the maximal torus subgroup $H = U(1)^{N-1}$ for $G = SU(N)$. This choice immediately determines the type of magnetic monopoles. We re-examine this issue. We have learned that the magnetic monopole which is responsible for area law of the Wilson loop is determined by the maximal stability group $\tilde{H}$ rather than the residual gauge group H. This is a new feature appeared in $SU(N), N \geq 3$. It seems that this possibility has been overlooked so far in the lattice community as far as I know. Indeed, this situation occurs only for $SU(N), N \geq 3$. Therefore, we must distinguish the maximal stability group $\tilde{H}$ from the residual gauge group H. In general, the maximal stability group $\tilde{H}$ is larger than the maximal torus subgroup, $H = U(1)^{N-1} \subset \tilde{H}$. So, the coset space is smaller than the maximal torus case, i.e., $G/\tilde{H} \subset G/H$.

The existence of magnetic monopole is suggested from the non-trivial Homotopy groups $\pi_2(G/H)$. In the case (II),

$$\pi_2(F_2) = \pi_2(SU(3)/(U(1) \times U(1))) = \pi_1(U(1) \times U(1)) = \mathbf{Z} + \mathbf{Z}. \tag{5.1}$$

On the other hand, in the case (I), i.e., [m,0] or [0,n]

$$\pi_2(CP^2) = \pi_2(SU(3)/U(2)) = \pi_1(U(2)) = \pi_1(SU(2) \times U(1)) = \pi_1(U(1)) = \mathbf{Z}. \tag{5.2}$$

Note that CP^{N-1} model has only the local $U(1)$ invariance for any $N \geq 2$. It is this U(1) invariance that corresponds to a single kind of Abelian magnetic monopole appearing in the case (I). This magnetic monopole may be related to the non-Abelian magnetic monopole proposed by E. Weinberg et al. [47]. The explicit solution for the magnetic monopole in SU(3) gauge theories were discussed in [48].

This situation should be compared with the $SU(2)$ case where the maximal stability group is always given by the maximal torus $H = U(1)$ irrespective of the representation. Therefore, the coset is given by

$$G/H = SU(2)/U(1) = F_1 = CP^1 \cong S^2 \cong SO(3) \tag{5.3}$$

and

$$\pi_2(SU(2)/U(1)) = \pi_2(F_1) = \pi_2(CP^1) = \mathbf{Z}, \tag{5.4}$$

for *arbitrary* representation.

For $SU(N)$, our results suggest that the fundamental quarks are to be confined when the maximal stability group $\tilde{H}$ is given by $\tilde{H} = U(N-1)$ and

$$\pi_2(G/\tilde{H}) = \pi_2(SU(N)/U(N-1)) = \pi_2(CP^{N-1}) = \mathbf{Z}, \tag{5.5}$$

while the adjoint quark is related to the maximal torus $\tilde{H} = U(1)^{N-1}$ and

$$\pi_2(G/H) = \pi_2(SU(N)/U(1)^{N-1}) = \pi_2(F_{N-1}) = \mathbf{Z}^{N-1}. \tag{5.6}$$

This observation is in sharp contrast with the conventional treatment in which the $(N-1)$ species of magnetic monopoles corresponding to the residual maximal torus group $U(1)^{N-1}$ of $G = SU(N)$ were taken into account on equal footing. In fact, the NAST derived in this article shows that the fundamental quark feels only the $U(1)$ which is embedded in the maximal stability group $U(N-1)$ as a magnetic monopole. This is a component along the highest-weight.

6 Abelian dominance in SU(N) gauge theory

6.1 APEGT as a low-energy effective theory

The Abelian dominance in $SU(N)$ Yang-Mills theory can be explained as follows. First of all, we adopt the maximal Abelian (MA) gauge. The MA gauge for $SU(N)$

is defined as follows. Consider the Cartan decomposition of $\mathcal{A}$ into the diagonal (H) and off-diagonal (G/H) pieces,

$$\mathcal{A}(x) = \mathcal{A}^A(x)T^A = a^i(x)H^i + A^a(x)T^a, \quad (A = 1, \cdots, N^2-1). \tag{6.1}$$

Especially, for $G = SU(3)$,

$$H^1 = \frac{\lambda^3}{2}, \quad H^2 = \frac{\lambda^8}{2}, \quad T^a = \frac{\lambda^a}{2} \; (a = 1,2,4,5,6,7). \tag{6.2}$$

The MA gauge is obtained by minimizing the functional of off-diagonal fields,

$$\mathcal{R} := \int d^4x \frac{1}{2} A^a_\mu(x) A^a_\mu(x) := \int d^4x \mathrm{tr}_{G/H}(\mathcal{A}_\mu(x)\mathcal{A}_\mu(x)), \tag{6.3}$$

under the gauge transformation. Under the infinitesimal gauge transformation Λ, $\mathcal{R}$ transforms as

$$\begin{aligned}
\delta_\Lambda \mathcal{R} &= \int d^4x A^a_\mu \delta_\Lambda A^a_\mu \\
&= \int d^4x A^a_\mu (\partial_\mu \Lambda^a + g f^{aij} a^i_\mu \Lambda^j + g f^{aib} a^i_\mu \Lambda^b + g f^{abC} A^b_\mu \Lambda^C) \\
&= -\int d^4x (\partial_\mu A^a_\mu + g f^{aib} a^i_\mu A^b_\mu)\Lambda^a,
\end{aligned} \tag{6.4}$$

since f^{ABC} is completely antisymmetric in the indices and $f^{aij} = 0$ (T^i and T^j are commutable). Therefore, the condition $\delta_\Lambda \mathcal{R} = 0$ for arbitrary Λ leads to the differential MA gauge given by

$$\partial_\mu A^a_\mu(x) - g f^{abi} a^i_\mu(x) A^b_\mu(x) := (D_\mu[a]A_\mu)^a = 0. \tag{6.5}$$

The $SU(N)$ Yang-Mills theory in MA gauge is given by

$$S^{total}_{YM} = S_{YM} + S_{GF}, \tag{6.6}$$

$$S_{YM} = \int d^4x \frac{1}{4} \mathcal{F}^A_{\mu\nu} \mathcal{F}^A_{\mu\nu}, \tag{6.7}$$

$$S_{GF} = \int d^4x i\delta_B \left[\bar{C}^a \left(D_\mu[a]A_\mu + \frac{\alpha}{2}B\right)^a\right] + \int d^4x i\delta_B \left[\bar{C}^i \left(\partial_\mu a_\mu + \frac{\beta}{2}B\right)^i\right], \tag{6.8}$$

where δ_B is the Becchi-Rouet-Stora-Tyupin (BRST) transformation,

$$\begin{aligned}
\delta_B \mathcal{A}_\mu(x) &= \mathcal{D}_\mu[\mathcal{A}]C(x) := \partial_\mu C(x) - ig[\mathcal{A}_\mu(x), C(x)], \\
\delta_B C(x) &= i\frac{1}{2} g[C(x), C(x)], \\
\delta_B \bar{C}(x) &= iB(x), \\
\delta_B B(x) &= 0,
\end{aligned} \tag{6.9}$$

which is nilpotent, i.e., $\delta_B^2 \equiv 0$. The generating functional is given by

$$Z_{YM}[J] = \int [d\mathcal{A}][dC][d\bar{C}][dB] \exp(-S^{total}_{YM} - S_J). \tag{6.10}$$

Now we proceed to derive the effective Abelian gauge theory in the MA gauge by integrating out the off-diagonal gauge fields (together with the ghost and anti-ghost fields), $A^a, C^a, \bar{C}^a, B^a$ [49, 12]. Then the $SU(N)$ Yang-Mills theory can be reduced to the $U(1)^{N-1}$ Abelian gauge theory which is written in terms of the diagonal fields alone, $a^i, C^i, \bar{C}^i, B^i$.

$$Z_{YM}[J] = \int [da^i][dC^i][d\bar{C}^i][dB^i] \exp(-S^{total}_{APEGT} - \tilde{S}_J), \tag{6.11}$$

where

$$\exp(-S^{total}_{APEGT} - \tilde{S}_J) = \int [dA^a][dC^a][d\bar{C}^a][dB^a] \exp(-S^{total}_{YM} - S_J). \tag{6.12}$$

In particular, the partition function reads

$$Z_{YM}[0] = \int [da^i][dC^i][d\bar{C}^i][dB^i] \exp(-S^{total}_{APEGT}) := Z_{APEGT}. \tag{6.13}$$

The Abelian gauge theory obtained in this way was called the Abelian-projected effective gauge theory (APEGT). It has been shown that the APEGT is an Abelian gauge theory with the gauge coupling constant g which runs according to the same renormalization-group beta function as the original $SU(N)$ Yang-Mills theory,

$$S^{total}_{APEGT}[a^i, C^i, \bar{C}^i, B^i] = \int d^4x \frac{1}{4g^2(\mu)}(da^i, da^i) + S'_{GF}, \tag{6.14}$$

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \frac{b_0}{8\pi^2} \ln \frac{\mu}{\mu_0}, \quad b_0 = \frac{11N}{3} > 0, \tag{6.15}$$

up to the one-loop level (see [12] for details), [11] Hence

$$\begin{aligned} \left\langle f[a^j] \right\rangle_{YM} &= Z^{-1}_{YM} \int [d\mathcal{A}][dC][d\bar{C}][dB] \exp(-S^{total}_{YM}) f[a^j] && (6.16) \\ &\cong Z^{-1}_{APEGT} \int [da^i][dC^i][d\bar{C}^i][dB^i] \exp(-S^{total}_{APEGT}) f[a^j] && (6.17) \\ &= \left\langle f[a^j] \right\rangle_{APEGT}. && (6.18) \end{aligned}$$

The APEGT obtained in this way can be regarded as the low-energy effective gauge theory of Yang-Mills theory. In the MA gauge, the off-diagonal gauge fields A^a_μ have the non-zero mass m_A, $m_A \neq 0$, whereas the diagonal gauge fields a^i_μ remain massless, $m_a = 0$. This was confirmed by Monte Carlo simulation on a lattice by Amemiya and Suganuma [51] for $G = SU(2)$ and $SU(3)$. The non-zero mass of the off-diagonal pieces was shown analytically at least in the topological sector, based on the dimensional reduction to the two-dimensional coset non-linear sigma model, see section IV.C of [13]. More details will be given in a subsequent article [42]. Therefore, integration of the massive off-diagonal gauge fields is interpreted as a step

[11] The result of [49, 12] for SU(2) can be generalized to SU(N) in the straightforward way, at least in one-loop level [50]. In the two-loop level, it is not trivial. The two-loop result will be given in [50].

of the Wilsonian renormalization group. [12] If so, the APEGT can describe the low-energy physics in the length scale $R > m_A^{-1}$. In this sense, the APEGT is regarded as the low-energy effective gauge theory of Yang-Mills theory.

6.2 Abelian dominance

By virtue of the NAST for $SU(N)$ just derived, Abelian dominance in $SU(N)$ Yang-Mills theory is explained as follows, based on the same argument as the SU(2) case.[13] The NAST (4.27) implies that the expectation value of the Wilson loop in the $SU(N)$ Yang-Mills theory is given by

$$\langle W^C[\mathcal{A}]\rangle_{YM} = \int [d\mu(V)]_C \Big\langle \exp\left(ig \oint_C a\right) \Big\rangle_{YM}, \tag{6.19}$$

where the one-form a is written as

$$a = 2\mathrm{tr}(\mathcal{H}\mathcal{A}^V), \tag{6.20}$$

for quark in the fundamental representation. So, the one-form a is equal to the diagonal piece of the gauge-transformed potential $\mathcal{A}^V$, i.e., a component along the highest-weight vector of the fundamental representation of $SU(N)$. [14] By applying the above result (6.18) to (6.19), we obtain

$$\begin{aligned} \Big\langle W^C[\mathcal{A}]\Big\rangle_{YM} &= \Big\langle \exp\left(ig \oint_C a\right) \Big\rangle_{YM} && (6.21) \\ &\cong \Big\langle \exp\left(ig \oint_C a\right) \Big\rangle_{APEGT} && (6.22) \\ &= Z_{APEGT}^{-1} \int [da^i][dC^i][d\bar{C}^i][dB^i] \exp(-S_{APEGT}^{total}[a^i]) \exp\left(ig \oint_C a\right), && (6.23) \end{aligned}$$

where we have used in the first equality the fact that the the integration over V is redundant after having taken the expectation value $\langle\cdot\rangle_{YM}$. This implies the Abelian dominance for the large Wilson loop in $SU(N)$ gauge theory in the sense that the expectation value of the non-Abelian Wilson loop in Yang-Mills theory is nearly equal to (or dominated by) that of the Abelian Wilson loop in the APEGT where the Wilson loop C is so large that the APEGT is valid in that region, i.e., . $R, T > m_A^{-1}$ for a rectangular Wilson loop with sides R and T. The APEGT is valid in the range $\Lambda_{QCD} < \mu \sim R^{-1} < m_A$.

[12] Rigorously speaking, all high-energy modes should be integrated out in the Wilsonian renormalization group. So, we must integrate out the high-energy mode of the diagonal fields a_μ^i too. The result is the same as the above, at least in the one-loop level. In the two-loop level, we must be more careful in dealing with the high-energy mode, see [50].

[13] The Abelian dominance in the low-energy region of SU(2) QCD was already shown in [15] based on the result [12] combined with the SU(2) NAST [21, 15]. The monopole dominance was derived for $SU(2)$ in [15] by showing that the dominant contribution to the area law comes from the monopole piece alone, $\Omega_K = d\omega = \mathrm{tr}(\mathbf{n}[d\mathbf{n}, d\mathbf{n}])$.

[14] Therefore, the component a along the highest-weight is obtained as an appropriate linear combination of a^i. Other components orthogonal to the highest-weight don't contribute to the expectation value of the Wilson loop, since the APEGT (6.14) is an Abelian gauge theory without self-interaction among the gauge fields.

6.3 Monopole dominance and area law

In our framework, the Abelian dominance and the monopole dominance are understood as implying the first and the second equality respectively,

$$\left\langle W^C[\mathcal{A}] \right\rangle_{YM} \cong \left\langle \exp\left(ig \oint_C a \right) \right\rangle_{APEGT} \tag{6.24}$$

$$\cong \left\langle \exp\left(i \oint_C \omega \right) \right\rangle_{APEGT}. \tag{6.25}$$

Numerical simulations show that the monopole part exhibits the area law and σ_{Abel} exhausts the full string tension obtained from the non-Abelian Wilson loop (i.e., monopole dominance in the string tension or area law),

$$\left\langle \exp\left(i \oint_C \omega \right) \right\rangle_{APEGT} \sim \exp(-\sigma_{Abel}|S|), \tag{6.26}$$

while $\left\langle \exp\left(ig \oint_C a - i \oint_C \omega \right) \right\rangle_{APEGT}$ does not exhibits the area law. This result implies that the area law of the original non-Abelian Wilson loop,

$$\left\langle W^C[\mathcal{A}] \right\rangle_{YM} \sim \exp(-\sigma|S|) \quad \sigma \cong \sigma_{Abel}. \tag{6.27}$$

In [16], the monopole dominance and the area law of the Wilson loop have been shown based on the APEGT for $G = SU(2)$. Now this scenario can be extended into $G = SU(N)$. The important remarks are in order,

1. The APEGT has a running coupling constant which increases as the relevant energy decreases (asymptotic freedom), so the APEGT is in the strong coupling region in the low-energy regime. See Fig.6

2. The Abelian gauge group in APEGT is a compact group embedded in the compact non-Abelian gauge group $SU(N)$. It is the compactness that causes the phase transition in the APEGT at

$$\alpha_c = \frac{\pi}{4}, \tag{6.28}$$

which separates the Coulomb (Conformal) phase ($\alpha < \alpha_c$) from the strong coupling phase ($\alpha > \alpha_c$). This follows from the Berezinski-Kosterlitz-Thouless (BKT) phase transition in the two-dimensional O(2) NLS model obtained by the dimensional reduction.

3. In the low-energy region such that $\alpha(\mu) > \alpha_c$, the APEGT is in the strong coupling phase which confines the quark due to vortex condensation. The above strategy is schematically shown as follows.

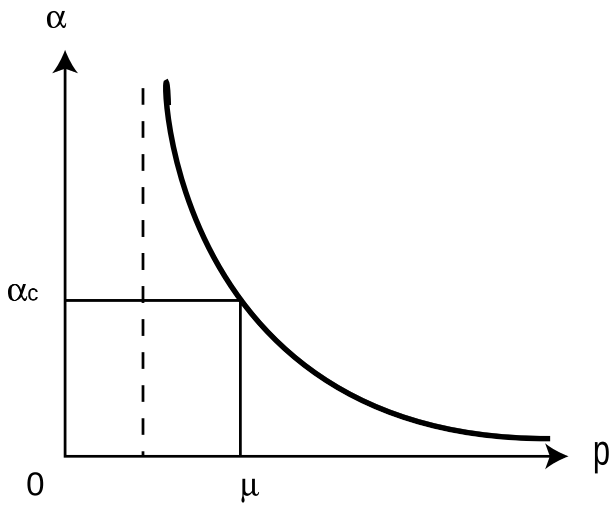
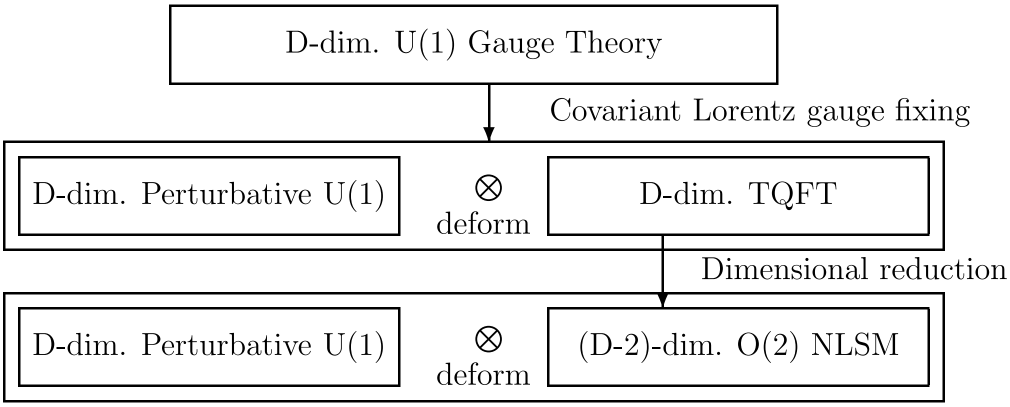


Figure 6: The running coupling constant in the APEGT.



7 Reformulation of Yang-Mills theory

In this section we summarize a novel reformulation of the Yang-Mills theory which was proposed in [13] and elaborated in [17]. This material is necessary in the subsequent sections.

7.1 Deformation of topological field theory

We consider the quantization of Yang-Mills theory on the topological background field. We decompose the connection $\mathcal{A}$ into

$$\mathcal{A}_\mu(x) = \Omega_\mu(x) + \mathcal{Q}_\mu(x), \tag{7.1}$$

and identify $\mathcal{Q}$ with the quantum fluctuation field on the background field Ω. For arbitrary but fixed background field Ω, the generating functional is given by

$$\begin{aligned} & \tilde{Z}[J,\Omega] \\ := & \int [d\mathcal{Q}][d\tilde{C}][d\bar{\tilde{C}}][d\tilde{B}] \exp\left\{ iS_{YM}[\Omega+\mathcal{Q}] + i\tilde{S}_{GF}[Q,\tilde{C},\bar{\tilde{C}},\tilde{B}] + i(J_\mu \cdot \mathcal{Q}_\mu)\right\} \end{aligned} \tag{7.2}$$

where $S_{YM}[\mathcal{A}]$ is the usual Yang-Mills action,

$$S_{YM}[\mathcal{A}] = -\int d^D x \frac{1}{4}\mathcal{F}_{\mu\nu}[\mathcal{A}]\mathcal{F}_{\mu\nu}[\mathcal{A}], \tag{7.3}$$

and $\tilde{S}_{GF}$ is the gauge-fixing and FP ghost term for the quantum fluctuation field $\mathcal{Q}$,

$$\tilde{S}_{GF}[\mathcal{Q},\tilde{C},\bar{\tilde{C}},\tilde{B}] \ := \ -\int d^D x \ i\tilde{\delta}_B \ \mathrm{tr}_G \left[\bar{\tilde{C}}\left(\tilde{F}[\mathcal{Q}] + \frac{\tilde{\alpha}}{2}\tilde{B}\right)\right]. \tag{7.4}$$

We wish to retain the gauge invariance for the background field Ω even after the gauge fixing for $\mathcal{Q}$. This is realized by choosing the background field (BGF) gauge fixing condition,

$$\tilde{F}^A[\mathcal{Q}] := \mathcal{D}_\mu^{AB}[\Omega]\mathcal{Q}_\mu^B = 0. \tag{7.5}$$

In fact, in the BGF gauge, $\tilde{Z}[J,\Omega]$ is invariant under the gauge transformation of the background field; the infinitesimal version is given by $\delta\Omega_\mu = \mathcal{D}_\mu[\Omega]\omega$. Hence, the theory with the action,

$$\tilde{S}_{eff}[J,\Omega] := -i \ln \tilde{Z}[J,\Omega], \tag{7.6}$$

is defined only on the space of gauge orbit. Suppose that the background field satisfies the equation, $F^A[\Omega]=0$. In order to consider the quantized Yang-Mills theory on all possible background fields satisfying the equations $F^A[\Omega]=0$, we define the total generating functional

$$\begin{aligned} & Z[J] \\ = & \int [d\Omega_\mu][dC][d\bar{C}][dB]\tilde{Z}[J,\Omega]\exp(iS_{TQFT}[\Omega,C,\bar{C},B])\exp[i(J_\mu\cdot\Omega_\mu)] \qquad (7.7) \\ = & \int [d\Omega_\mu][dC][d\bar{C}][dB]\exp\{i\tilde{S}_{eff}[J,\Omega] + iS_{TQFT}[\Omega,C,\bar{C},B] + i(J_\mu\cdot\Omega_\mu)\} \qquad (7.8) \end{aligned}$$

where $S_{TQFT}[\Omega,C,\bar{C},B]$ corresponds to the gauge-fixing term for the background field Ω_μ. In order to describe the magnetic monopole as a topological background field in Yang-Mills theory, we choose the MA gauge for Ω_μ, i.e.,

$$S_{TQFT}[\Omega,C,\bar{C},B] \ := \ -\int d^D x \ i\delta_B \ \mathrm{tr}_{G/H}\left[\bar{C}\left(F[\Omega] + \frac{\alpha}{2}B\right)\right], \tag{7.9}$$

where

$$F^a[\Omega] := D_\mu{}^{ab}[\Omega^i]\Omega_\mu^b := (\partial_\mu\delta^{ab} - gf^{abi}\Omega_\mu{}^i)\Omega_\mu^b \quad (i=1,\cdots,N-1). \tag{7.10}$$

Note that the trace is taken on the coset G/H, not on the entire G, in the MA gauge.

Under the identification,

$$\Omega_\mu(x) := \frac{i}{g}U(x)\partial_\mu U^\dagger(x), \quad \mathcal{Q}_\mu(x) := U(x)\mathcal{V}_\mu(x)U^\dagger(x), \tag{7.11}$$

we assume that all the topologically non-trivial configurations come from Ω, whereas $\mathcal{V}$ denotes the topologically trivial configurations. Therefore, $\mathcal{V}$ changes under the small gauge transformation, while Ω includes the effect of large gauge transformations, so we must take into account the finite gauge rotation U, without restricting to the infinitesimal gauge transformation. Under this identification (7.11), the Yang-Mills action is invariant,

$$S_{YM}[\mathcal{A}] = S_{YM}[\Omega + \mathcal{Q}] = S_{YM}[\mathcal{V}] = \int d^Dx \frac{1}{4}\mathcal{F}_{\mu\nu}[\mathcal{V}]\mathcal{F}_{\mu\nu}[\mathcal{V}], \tag{7.12}$$

while the gauge-fixing term (7.4) is changed [17] into

$$\tilde{S}_{GF}[\mathcal{V},\gamma,\bar{\gamma},\beta] \;\; := \;\; -\int d^Dx \; i\tilde{\delta}_B \; \mathrm{tr}_G\left[\bar{\gamma}\left(\partial_\mu\mathcal{V}_\mu + \frac{\tilde{\alpha}}{2}\beta\right)\right]. \tag{7.13}$$

This implies that the background gauge for $\mathcal{Q}$ is changed into the Lorentz gauge for $\mathcal{V}$, $\partial_\mu\mathcal{V}_\mu = 0$. Then the generating functional in the background field Ω is cast into

$$\tilde{Z}[J,\Omega] = \int [d\mathcal{V}][d\gamma][d\bar{\gamma}][d\beta] \exp\left\{i[S_{YM}[\mathcal{V}] + \tilde{S}_{GF}[\mathcal{V},\gamma,\bar{\gamma},\beta] + (J_\mu \cdot U\mathcal{V}_\mu U^\dagger)]\right\} \tag{7.14}$$

where $\mathcal{V}_\mu, \gamma, \bar{\gamma}, \beta$ are defined by the adjoint rotation of $\mathcal{Q}_\mu, \tilde{C}, \bar{\tilde{C}}, \bar{B}$ respectively,

$$\mathcal{V}_\mu := U^\dagger\mathcal{Q}_\mu U, \quad \gamma := U^\dagger\tilde{C}U, \quad \bar{\gamma} := U^\dagger\bar{\tilde{C}}U, \quad \beta := U^\dagger\tilde{B}U. \tag{7.15}$$

Thus the total generating functional reads

$$\begin{aligned} & Z[J] \\ = & \int [dU][dC][d\bar{C}][dB]\tilde{Z}[J,\Omega]\exp(iS_{GF}[\Omega,C,\bar{C},B])\exp[i(J_\mu\cdot\Omega_\mu)] \quad (7.16) \\ = & \int [dU][dC][d\bar{C}][dB]\exp\{i\tilde{S}_{eff}[J,\Omega] + iS_{TQFT}[\Omega,C,\bar{C},B] + i(J_\mu\cdot\Omega_\mu)\}, \quad (7.17) \end{aligned}$$

where we have made the change of variable from Ω to U. The measure $[dU]$ is invariant under the multiplication,

$$U(x) \to \tilde{U}(x)U(x), \tag{7.18}$$

which leads to the finite gauge transformation of the background field,

$$\Omega(x) \to \tilde{U}(x)\Omega(x)\tilde{U}^\dagger(x) + \frac{i}{g}\tilde{U}(x)d\tilde{U}^\dagger(x). \tag{7.19}$$

It is more efficient to modify the MA gauge into the OSp(D/2) invariant form [13],

$$S_{TQFT}[\Omega,C,\bar{C},B] := \int_{\mathbf{R}^D} d^Dx \; i\delta_B\bar{\delta}_B \mathrm{tr}_{G/H}\left[\frac{1}{2}\Omega_\mu(x)\Omega_\mu(x) + iC(x)\bar{C}(x)\right], \tag{7.20}$$

by making use of the BRST δ_B and anti-BRST $\bar{\delta}_B$ transformations, see [13] for details.[15]

The expectation value of the functional $f(\mathcal{A})$ of $\mathcal{A}$ is calculated as follows. If it is in the form, $f(\mathcal{A}) = g(\mathcal{V}_\mu, U)h(U)$, then

$$\langle f(\mathcal{A})\rangle_{YM} = \langle\langle g(\mathcal{V}_\mu, U)h(U)\rangle^{\mathcal{V}}_{pYM}\rangle^{U}_{TQFT} = \langle\langle g(\mathcal{V}_\mu, U)\rangle^{\mathcal{V}}_{pYM} h(U)\rangle^{U}_{TQFT}, \qquad (7.21)$$

where the sector of the perturbative deformation is defined by

$$\langle(\cdots)\rangle^{\mathcal{V}}_{pYM} = Z^{-1}_{pYM}\int [d\mathcal{V}][d\gamma][d\bar{\gamma}][d\beta] \exp\left\{i(S_{YM}[\mathcal{V}] + \tilde{S}_{GF}[\mathcal{V},\gamma,\bar{\gamma},\beta])\right\}(\cdots), \qquad (7.22)$$

$$Z_{pYM} := \int [d\mathcal{V}][d\gamma][d\bar{\gamma}][d\beta] \exp\left\{i(S_{YM}[\mathcal{V}] + \tilde{S}_{GF}[\mathcal{V},\gamma,\bar{\gamma},\beta])\right\}, \qquad (7.23)$$

and the sector topological field theory is defined by

$$\langle(\cdots)\rangle^{U}_{TQFT} := Z^{-1}_{TQFT}\int [dU][dC][d\bar{C}][dB] \exp\{iS_{TQFT}[\Omega, C, \bar{C}, B]\}(\cdots), \qquad (7.24)$$

$$Z_{TQFT} := \int [dU][dC][d\bar{C}][dB] \exp\{iS_{TQFT}[\Omega, C, \bar{C}, B]\}. \qquad (7.25)$$

This reformulation of the Yang-Mills theory was called the perturbative deformation of a topological quantum field theory. The expectation value $\langle(\cdots)\rangle^{\mathcal{V}}_{pYM}$ for the field $\mathcal{V}$ is calculated in the perturbation theory in the coupling constant g. On the other hand, the expectation value $\langle(\cdots)\rangle^{U}_{TQFT}$ should be calculated in a non-perturbative way to incorporate the topological contribution. Here U is a compact gauge variable corresponding to finite gauge transformation. In the instanton calculus, the integration measure $[dU]$ is replaced with the (finite-dimensional) integration over the collective coordinates of the instanton.

7.2 Dimensional reduction to NLS model in the MA gauge

It is shown [13] that, due to the hidden supersymmetry, OSp(D/2), the TQFT part (7.20) is reduced to the $(D-2)$-dimensional coset (G/H) nonlinear sigma (NLS) model,

$$S_{NLSM}[U, C, \bar{C}] := 2\pi \int_{\mathbf{R}^{D-2}} d^{D-2}x \ \mathrm{tr}_{G/H}\left[\frac{1}{2}\Omega_\mu(x)\Omega_\mu(x) + iC(x)\bar{C}(x)\right], \qquad (7.26)$$

where for the matrix element Ω_{ab} (see Appendix C)

$$\mathrm{tr}_{G/H}\left[\frac{1}{2}\Omega_\mu(x)\Omega_\mu(x)\right] = \sum_{a,b:a<b} 2(\Omega_\mu(x))_{ab}(\Omega_\mu(x))_{ab}. \qquad (7.27)$$

[15] Note that (7.20) is obtained from (7.9) by adding the ghost self-interaction terms and choosing $\alpha = -2$, since

$$-\bar{\delta}_B\left[\frac{1}{2}\Omega^a_\mu(x)\Omega^a_\mu(x) + iC^a(x)\bar{C}^a(x)\right] = \bar{C}^a(D_\mu[\Omega^i]\Omega_\mu - B)^a + \bar{C}^a[C,\bar{C}]^a - \frac{1}{2}C^a[\bar{C},\bar{C}]^a.$$

The last two terms reduce to $2i\bar{c}^1\bar{c}^2c^3 = -2c^3\bar{c}^+\bar{c}^-$ for $G = SU(2)$ in agreement with the result [13].

By making use of the complex coordinates on the flag space G/H, the action is rewritten as (see Appendix C)

$$S_{NLSM} = \frac{4\pi}{g^2}\int_{\mathbf{R}^{D-2}} d^{D-2}x g_{\alpha\bar{\beta}} \frac{\partial w^\alpha}{\partial x_a}\frac{\partial \bar{w}^\beta}{\partial x_a} \quad (a = 1, \cdots, D-2), \tag{7.28}$$

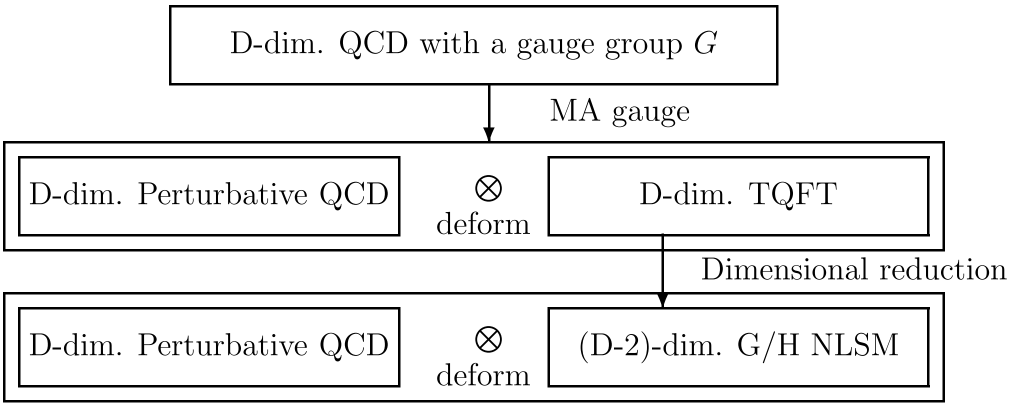
where we have omitted to write the decoupled ghost term, $C(x)\bar{C}(x)$. Especially, for $D = 4$,

$$S_{NLSM} = \frac{4\pi}{g^2}\int_{\mathbf{C}} dz d\bar{z}\ g_{\alpha\bar{\beta}} \left(\frac{\partial w^\alpha}{\partial z}\frac{\partial \bar{w}^\beta}{\partial \bar{z}} + \frac{\partial w^\alpha}{\partial \bar{z}}\frac{\partial \bar{w}^\beta}{\partial z} \right), \tag{7.29}$$

where $z = x + iy = x_1 + ix_2 \in \mathbf{C} \cong \mathbf{R}^2$ and $dxdy = dx_1 dx_2 = \frac{i}{2} dz d\bar{z}$. The $G = SU(2)$ case was already analyzed in [13]

$$S_{NLSM} = \frac{4\pi}{g^2}\int_{\mathbf{C}} dz d\bar{z}\ \frac{1}{(1+|w|^2)^2} \left(\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial z} \right). \tag{7.30}$$

The above strategy is schematically shown as follows.



8 Area law of the Wilson loop (I)

In this section we derive the area law of the Wilson loop based on the instanton calculus [52, 53, 54]. More systematic estimation will be given in the next section based on the large N expansion.

The static potential $V(R)$ is evaluated from the rectangular Wilson loop C with sides T and R according to

$$V(R) := -\lim_{T\to\infty} \frac{1}{T} \ln \langle W^C[\mathcal{A}] \rangle_{YM_4}. \tag{8.1}$$

The (full) string tension σ is defined by

$$\sigma := -\lim_{A(C)\to\infty} \frac{1}{A(C)} \ln \langle W^C[\mathcal{A}] \rangle_{YM_4}, \tag{8.2}$$

where $A(C)$ is the minimal area of the surface spanned by the Wilson loop C. Of course, the rectangular loop has the minimal area, $A(C) = TR$.

Using the NAST for the Wilson loop operator,

$$W^C[\mathcal{A}] = \int [d\mu(\xi)]_C \exp\left(ig \oint_C n^A \mathcal{A}^A + i \oint_C \omega\right), \tag{8.3}$$

we can write its expectation value in the Yang-Mills theory as [15, 17]

$$\begin{aligned} &\langle W^C[\mathcal{A}]\rangle_{YM_4} \\ = &\left\langle\left\langle \exp\left[ig \oint_C dx^\mu n^A(x)\mathcal{V}_\mu^A(x)\right]\right\rangle_{pYM_4} \exp\left[i\oint_C \omega\right]\right\rangle_{TQFT_4}. \end{aligned} \tag{8.4}$$

8.1 Perturbative expansion and dimensional reduction

In our reformulation, the field $\mathcal{V}_\mu^A(x)$ is identified with the perturbative deformation, so we expand the first exponential of (8.4) in powers of the coupling constant g:

$$\begin{aligned} &\left\langle \exp\left[ig \oint_C dx^\mu n^A(x)\mathcal{V}_\mu^A(x)\right]\right\rangle_{pYM_4} \\ = \ & 1 + \sum_{n=1}^\infty \frac{(ig)^n}{n!} \oint_C dx_1^{\mu_1} \oint_C dx_2^{\mu_2} \cdots \oint_C dx_n^{\mu_n} n^{A_1}(x_1) n^{A_2}(x_2)\cdots n^{A_n}(x_n) \\ &\times \langle \mathcal{V}_{\mu_1}^{A_1}(x_1)\mathcal{V}_{\mu_2}^{A_2}(x_2)\cdots \mathcal{V}_{\mu_n}^{A_n}(x_n)\rangle_{pYM_4}. \end{aligned} \tag{8.5}$$

Hence we obtain

$$\begin{aligned} &\langle W^C[\mathcal{A}]\rangle_{YM_4} \\ =&\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{TQFT_4} + \sum_{n=1}^\infty \frac{(ig)^n}{n!} \oint_C dx_1^{\mu_1} \oint_C dx_2^{\mu_2} \cdots \oint_C dx_n^{\mu_n} \\ &\times\left\langle n^{A_1}(x_1) n^{A_2}(x_2)\cdots n^{A_n}(x_n) \exp\left[i\oint_C \omega\right]\right\rangle_{TQFT_4} \\ &\times \langle \mathcal{V}_{\mu_1}^{A_1}(x_1)\mathcal{V}_{\mu_2}^{A_2}(x_2)\cdots \mathcal{V}_{\mu_n}^{A_n}(x_n)\rangle_{pYM_4}. \end{aligned} \tag{8.6}$$

$$\begin{aligned} = \ &\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{TQFT_4} \\ &\times\Bigg[1 + \sum_{n=1}^\infty \frac{(ig)^n}{n!} \oint_C dx_1^{\mu_1} \oint_C dx_2^{\mu_2} \cdots \oint dx_n^{\mu_n} \langle \mathcal{V}_{\mu_1}^{A_1}(x_1)\mathcal{V}_{\mu_2}^{A_2}(x_2)\cdots \mathcal{V}_{\mu_n}^{A_n}(x_n)\rangle_{pYM_4} \\ &\times \frac{\langle n^{A_1}(x_1) n^{A_2}(x_2)\cdots n^{A_n}(x_n) \exp[i\oint_C \omega]\rangle_{TQFT_4}}{\langle \exp[i\oint_C \omega]\rangle_{TQFT_4}}\Bigg]. \end{aligned} \tag{8.7}$$

We restrict the Wilson loop to the planar loop. This choice has the following advantage. For the planar Wilson loop C, the Parisi-Sourlas dimensional reduction leads to the following identities [13]:

$$\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{TQFT_4} = \left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{NLSM_2}, \tag{8.8}$$

and

$$\left\langle n^{A_1}(x_1)\cdots n^{A_n}(x_n)\exp\left[i\oint_C \omega\right]\right\rangle_{TQFT_4} = \left\langle n^{A_1}(x_1)\cdots n^{A_n}(x_n)\exp\left[i\oint_C \omega\right]\right\rangle_{NLSM_2}, \tag{8.9}$$

where $x_1, \cdots, x_n \in C \subset \mathbf{R}^2$. This is because, in the case of fundamental representation of $SU(N)$, n^A and ω are written in terms of U as (see (4.28), (4.29))

$$\begin{aligned} n^A(x) &= U_{1a}(x)(T^A)_{ab}\bar{U}_{1b}(x) = \bar{\phi}_a(x)(T^A)_{ab}\phi_b(x), && (8.10)\\ \omega(x) &= \frac{i}{2}[U_{1a}(x)d\bar{U}_{1a}(x) - dU_{1a}(x)\bar{U}_{1a}(x)] \\ &= \frac{i}{2}[\bar{\phi}_a(x)d\phi_a(x) - d\bar{\phi}_a(x)\phi_a(x)]. && (8.11)\end{aligned}$$

Taking the logarithm of the Wilson loop (8.7) and expanding it in powers of the coupling constant, we obtain

$$\begin{aligned} & \ln\langle W^C[\mathcal{A}]\rangle_{YM_4} && (8.12)\\ = & \ln\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{NLSM_2} \\ & + \ln\left[1 - \frac{g^2}{2}\oint_C dx^\mu \oint_C dy^\nu G^{AB}_{\mu\nu}(x,y)\frac{\langle n^A(x)n^B(y)\exp\left[i\int_S \Omega_K\right]\rangle_{NLSM_2}}{\langle\exp\left[i\oint_C \omega\right]\rangle_{NLSM_2}} + O(g^4)\right] \\ = & \ln\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{NLSM_2} \\ & -\frac{g^2}{2}\oint_C dx^\mu \oint_C dy^\nu G^{AB}_{\mu\nu}(x,y)\frac{\langle n^A(x)n^B(y)\exp\left[i\oint_C \omega\right]\rangle_{NLSM_2}}{\langle\exp\left[i\oint_C \omega\right]\rangle_{NLSM_2}} + O(g^4), && (8.13)\end{aligned}$$

where we have defined the two-point function,

$$G^{AB}_{\mu\nu}(x,y) := \langle \mathcal{V}^A_\mu(x)\mathcal{V}^B_\nu(y)\rangle_{pYM_4}. \tag{8.14}$$

In the rest of this section we focus on the first term in (8.13). The remaining terms will be estimated in the next section.

8.2 Instanton in F_{N-1} and CP^{N-1} models

We wish to demonstrate the area law for the expectation value,

$$\begin{aligned} & \left\langle \exp\left(i\oint_C \omega\right)\right\rangle_{NLSM_2} \equiv \left\langle \exp\left(i\int_S \Omega_K\right)\right\rangle_{NLSM_2} && (8.15)\\ = & Z^{-1}_{NLSM_2}\int [d\mu(w,\bar{w})]\exp(-S_{NLSM_2}[w,\bar{w}])\exp\left(i\int_S \Omega_K\right), && (8.16)\end{aligned}$$

where the two-dimensional NLS model is defined by the action,

$$\begin{aligned} S_{NLSM_2} &= \frac{4\pi}{g^2(\mu)}\int_{\mathbf{R}^2} d^2x g_{\alpha\bar{\beta}}\frac{\partial w^\alpha}{\partial x_a}\frac{\partial \bar{w}^\beta}{\partial x_a} && (8.17)\\ &= \frac{4\pi}{g^2(\mu)}\int_{\mathbf{C}} dzd\bar{z}\ g_{\alpha\bar{\beta}}\left(\frac{\partial w^\alpha}{\partial z}\frac{\partial \bar{w}^\beta}{\partial \bar{z}} + \frac{\partial w^\alpha}{\partial \bar{z}}\frac{\partial \bar{w}^\beta}{\partial z}\right), && (8.18)\end{aligned}$$

where

$$z = x + iy = x_1 + ix_2 \in \mathbf{C} \cong \mathbf{R}^2, \quad dxdy = dx_1 dx_2 = \frac{i}{2} dz d\bar{z}, \tag{8.19}$$

and $g(\mu)$ is the Yang-Mills coupling constant.[16]

Note that the action satisfies the inequality,

$$\begin{aligned} S_{NLSM} &= \frac{2\pi}{g^2(\mu)} \int_{\mathbf{R}^2} d^2x g_{\alpha\bar{\beta}} (\partial_a w^\alpha \pm i\epsilon_{ab}\partial_b w^\alpha)(\partial_a w^\beta \pm i\epsilon_{ac}\partial_c w^\beta)^* \\ &\pm i \frac{4\pi}{g^2(\mu)} \int_{\mathbf{R}^2} d^2x \epsilon_{ab} g_{\alpha\bar{\beta}} \frac{\partial w^\alpha}{\partial x_a} \frac{\partial \bar{w}^\beta}{\partial x_b} \qquad (8.20) \\ &\geq \pm i \frac{4\pi}{g^2(\mu)} \int_{\mathbf{R}^2} d^2x \epsilon_{ab} g_{\alpha\bar{\beta}} \frac{\partial w^\alpha}{\partial x_a} \frac{\partial \bar{w}^\beta}{\partial x_b}. \qquad (8.21) \end{aligned}$$

This inequality is saturated when w^α satisfies the equation $\partial_a w^\alpha \pm i\epsilon_{ab}\partial_b w^\alpha = 0$, which is equivalent to the Cauchy-Riemann equation,

$$\bar{\partial}_z w^\alpha := (\partial_1 + i\partial_2) w^\alpha = 0. \tag{8.22}$$

The solution $w^\alpha = f^\alpha(z)$ is an arbitrary rational function of z.

The finite action configuration of the coset NLS model is provided by the instanton solution, which is a solution of the Cauchy-Riemann equation (8.22). It is known [23] that the integer-valued topological charge Q of the instanton in the F_{N-1} NLS model is given by the integral of the Kähler 2-form over $\mathbf{R}^2$,

$$Q = \frac{1}{\pi} \int \Omega_K = \int_{\mathbf{R}^2} \frac{d^2x}{\pi} \epsilon_{ab} g_{\alpha\bar{\beta}} \frac{\partial w^\alpha}{\partial x_a} \frac{\partial \bar{w}^\beta}{\partial x_b} = \int_{\mathbf{C}} \frac{dz d\bar{z}}{\pi} \, g_{\alpha\bar{\beta}} \left(\frac{\partial w^\alpha}{\partial z} \frac{\partial \bar{w}^\beta}{\partial \bar{z}} - \frac{\partial w^\alpha}{\partial \bar{z}} \frac{\partial \bar{w}^\beta}{\partial z} \right) \tag{8.23}$$

This is a generalization of the F_1 case ($N = 2$),

$$Q = \frac{i}{2\pi} \int_{\mathbf{C}} \frac{dw d\bar{w}}{(1 + |w|^2)^2} = \frac{i}{2\pi} \int_{S^2} \frac{dz d\bar{z}}{(1 + |w|^2)^2} \left(\frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} - \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right). \tag{8.24}$$

Thus the instanton solution is characterized by the integral topological charge $Q \in \mathbf{Z}$. For instanton ($Q > 0$) and anti-instanton ($Q < 0$) configurations with a topological charge Q, the action has

$$S_{NLSM} = \frac{4\pi^2}{g^2} |Q|. \tag{8.25}$$

The metric in the Kähler manifold F_{N-1} is obtained according to $g_{\alpha\bar{\beta}} = \frac{\partial}{\partial w^\alpha} \frac{\partial}{\partial \bar{w}^\beta} K$ from the the Kähler potential,

$$K(w, \bar{w}) = \sum_{\ell=1}^{N-1} d_\ell K_\ell(w, \bar{w}) = \sum_{\ell=1}^{N-1} d_\ell \ln \Delta_\ell(w, \bar{w}), \tag{8.26}$$

[16] The running of the coupling constant will be given by the perturbative deformation in four-dimensional Yang-Mills theory as in (6.15).

with the Dynkin indices $d_\ell(\ell = 1, \cdots, N-1)$. Then the integral $\int \Omega_K$ over the whole two-dimensional space reads

$$\int_{\mathbf{R}^2} \Omega_K = \pi Q = \pi \sum_{\ell=1}^{N-1} d_\ell Q_\ell, \tag{8.27}$$

where Q_ℓ are integers-valued topological charges. Hence, $\Omega_K(x)/\pi$ is identified with the density of the topological charge (up to the weight due to the index d_ℓ).

Now we consider the CP^{N-1} model. If we identify w_α with the inhomogeneous coordinates, e.g., $w_\alpha := \frac{\phi_\alpha}{\phi_N}$, the metric (3.63) is rewritten as

$$g_{\alpha\bar{\beta}}(\phi) = \frac{(||\phi||^2)\delta_{\alpha\beta} - \bar{\phi}_\alpha \phi_\beta}{(||\phi||^2)^2}, \tag{8.28}$$

where

$$||\phi||^2 := \sum_{\alpha=1}^{N} |\phi_\alpha|^2 = |\phi_N|^2 \left(1 + |||w|||^2\right). \tag{8.29}$$

The action of CP^{N-1} model is given by

$$S_{CP^{N-1}} = \frac{4\pi}{g^2} \int d^d x g_{\alpha\bar{\beta}}(w) \partial_\mu w^\alpha \partial_\mu \bar{w}^\beta, \tag{8.30}$$

or equivalently,

$$S_{CP^{N-1}} = \frac{4\pi}{g^2} \int d^d x g_{\alpha\bar{\beta}}(\phi) \partial_\mu \phi^\alpha \partial_\mu \bar{\phi}^\beta. \tag{8.31}$$

Under the constraint $||\phi||^2 = 1$, the action is written as

$$\begin{aligned} S_{CP^{N-1}} &= \frac{4\pi}{g^2} \int d^d x (\delta_{\alpha\beta} - \bar{\phi}_\alpha \phi_\beta) \partial_\mu \phi^\alpha \partial_\mu \bar{\phi}^\beta \\ &= \frac{4\pi}{g^2} \int d^d x [(\partial_\mu \bar{\phi}^\alpha \partial_\mu \phi^\alpha + (\bar{\phi}^\alpha \partial_\mu \phi^\alpha)(\bar{\phi}^\beta \partial_\mu \phi^\beta)]. \end{aligned} \tag{8.32}$$

This agrees with the action of the CP^{N-1} model presented in [13], see Appendix C for more details.

8.3 Area law in the dilute instanton-gas approximation

If the Wilson loop is large compared with the typical size of the instanton, $\int_S \Omega_K(x)/\pi$ in (8.16) counts the number of instantons n_+^{in} minus anti-instantons n_-^{in} which are contained inside the area $S \subset \mathbf{R}^2$ bounded by the loop C, i.e.,

$$\int_S \Omega_K = \pi(n_+^{in} - n_-^{in}). \tag{8.33}$$

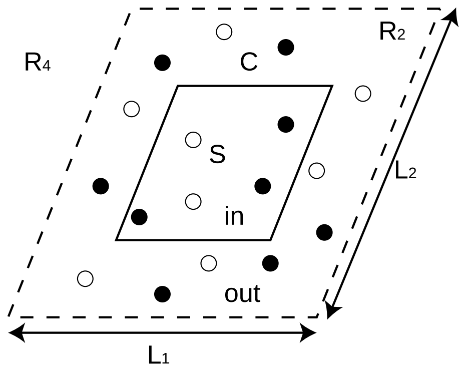


Figure 7: Instanton and anti-instanton configuration and the Wilson loop C in a finite-volume region $L_1 \times L_2$ in the two-dimensional plane $\mathbf{R}^2$ embedded in $\mathbf{R}^4$.

See Fig.7. Thus, the expectation value $\left\langle \exp\left(i \int_S \Omega_K\right)\right\rangle_{NLSM}$ is calculated by summing over all the possible cases of instanton and anti-instanton configurations (i.e., by the integration over the instanton moduli). In this calculation, we use

$$S_{NLSM} = \frac{4\pi^2}{g^2}|Q| = \frac{4\pi^2}{g^2}(n_+ + n_-), \quad n_\pm = n_\pm^{in} + n_\pm^{out}. \tag{8.34}$$

where n_+^{out} (n_-^{out}) is the number of instantons (anti-instantons) outside S and n_+ (n_-) is the total number of instantons (anti-instantons). For the quark in the fundamental representation $\mathbf{N}$ ($d_1 = 1, d_2 = d_3 = \cdots = d_{N-1} = 0$), this is easily performed as follows.

For the $SU(3)$ case with $[1,0]$, an element $\xi \in F_2$ is independent of w_3, so that w_3 is redundant in this case. Hence, it suffices to consider the CP^2 model for the fundamental quark (up to Weyl symmetry). For CP^2, the Kähler two-form is given by (4.87),

$$\begin{aligned}\Omega_K &= im(\Delta_1)^{-2}[(1+|w_1|^2)dw_2 \wedge d\bar{w}_2 - \bar{w}_2 w_1 dw_2 \wedge d\bar{w}_1 \\ &\quad -w_2\bar{w}_1 dw_1 \wedge d\bar{w}_2 + (1+|w_2|^2)dw_1 \wedge d\bar{w}_1].\end{aligned} \tag{8.35}$$

When $w_2 = 0$, Ω_K reduces to

$$\Omega_K = i(1+|w_1|^2)^{-2}dw_1 \wedge d\bar{w}_1. \tag{8.36}$$

Similarly, when $w_1 = 0$,

$$\Omega_K = i(1+|w_2|^2)^{-2}dw_2 \wedge d\bar{w}_2. \tag{8.37}$$

For a polynomial $w_\alpha = w_\alpha(z)$ in $z = x + iy$ with an order n, we find an instanton charge,

$$\int \Omega_K = \pi Q, \quad Q \in \mathbf{Z}. \tag{8.38}$$

This is the same situation as that encountered in $SU(2)$,

$$\int \Omega_K = 2j\pi Q \tag{8.39}$$

where $j = 1/2$ corresponds to the fundamental representation. [13, 15]. Thus the Wilson loop is estimated by the naive instanton calculus. In fact, the dilute instanton gas approximation leads to the area law for the Wilson loop, see [13]. Here the factor π is very important. The integral of Kähler two-form Ω_K is multiple of π. If we had a factor 2π, the area law was lost, just as the $j = 1$ case of $SU(2)$.

For the $SU(N)$ case with Dynkin index, $[1, 0, \cdots, 0]$, it suffices to consider the CP^{N-1} model. When $w_a \neq 0$ and $w_b = 0$ for all $b \neq a$, the Kähler two-form (4.53) for CP^{N-1} reduces to

$$\Omega_K = i(1 + |w_a|^2)^{-2} dw_a \wedge d\bar{w}_a, \tag{8.40}$$

with no summation over a. Therefore the above argument can be applied to any N of $SU(N)$. This implies confinement of fundamental quarks in $SU(N)$ Yang-Mills theory within the approximation of dilute instanton gas. This naive instanton calculation can be improved by including fluctuations from the instanton solutions following [54] and this issue will be discussed in detail in the subsequent article [42].

9 Area law of the Wilson loop (II)

The derivation of the area law of the Wilson loop in the four-dimensional Yang-Mills theory in the MA gauge is reduced to showing area law of the diagonal Wilson loop in the two-dimensional coset NLS model. In this section we complete a derivation of area law of the Wilson loop in the fundamental representation. In this section we use the large N expansion [55, 56, 57, 58, 59] for the coset NLS model, see e.g. [28, 29, 30, 31] for reviews of large N expansion.

To perform the large N expansion, it is convenient to introduce the new variables,

$$\begin{aligned} P_{ab}(x) &:= \bar{\phi}_a(x)\phi_b(x) = U_{1a}(x)\bar{U}_{1b}(x), & (9.1) \\ \mathcal{V}_\mu^{ab}(x) &:= \mathcal{V}_\mu^A(x)(T^A)_{ab}, & (9.2) \end{aligned}$$

which are used to rewrite

$$C_\mu(x) := n^A(x)\mathcal{V}_\mu^A(x) = P_{ab}(x)\mathcal{V}_\mu^{ab}(x), \tag{9.3}$$

where $A = 1, \cdots, N^2 - 1$ and $a, b = 1, \cdots N$ for $SU(N)$.

In terms of the above variables, the expansion (8.7) of the expectation value of the Wilson loop operator in powers of the coupling constant g is rewritten into

$$\begin{aligned} \langle W^C[\mathcal{A}] \rangle_{YM_4} &= \left\langle \exp\left[i \oint_C \omega \right] \right\rangle_{TQFT_4} + \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} \oint_C dx_1^{\mu_1} \oint_C dx_2^{\mu_2} \cdots \oint_C dx_n^{\mu_n} \\ &\times \langle \mathcal{V}_{\mu_1}^{a_1 b_1}(x_1) \mathcal{V}_{\mu_2}^{a_2 b_2}(x_2) \cdots \mathcal{V}_{\mu_n}^{a_n b_n}(x_n) \rangle_{pYM_4} \\ &\times \left\langle P_{a_1 b_1}(x_1) P_{a_2 b_2}(x_2) \cdots P_{a_n b_n}(x_n) \exp\left[i \oint_C \omega \right] \right\rangle_{TQFT_4} \end{aligned} \tag{9.4}$$

$$
\begin{aligned}
= & \left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{TQFT_4} \\
& \times\Bigg[1+\sum_{n=1}^{\infty}\frac{(ig)^n}{n!}\oint_C dx_1^{\mu_1}\oint_C dx_2^{\mu_2}\cdots\oint_C dx_n^{\mu_n} \\
& \times\langle \mathcal{V}_{\mu_1}^{a_1b_1}(x_1)\mathcal{V}_{\mu_2}^{a_2b_2}(x_2)\cdots\mathcal{V}_{\mu_n}^{a_nb_n}(x_n)\rangle_{pYM_4} \\
& \times\frac{\langle P_{a_1b_1}(x_1)P_{a_2b_2}(x_2)\cdots P_{a_nb_n}(x_n)\exp\left[i\oint_C\omega\right]\rangle_{TQFT_4}}{\langle \exp\left[i\oint_C\omega\right]\rangle_{TQFT_4}}\Bigg].
\end{aligned}
\tag{9.5}
$$

The diagrams needed to calculate this expectation value are drawn in Fig.9 based on the Feynmann rule given in Fig.8. Here it should be remarked that the definition of the Wilson loop operator

$$
\langle W^C[\mathcal{A}]\rangle := \frac{1}{\mathcal{N}}\left\langle \mathrm{tr}\left[\mathcal{P}\exp\left(ig\oint_C dx^\mu \mathcal{A}_\mu(x)\right)\right]\right\rangle_{YM_4}, \tag{9.6}
$$

includes the normalization factor $\mathcal{N}^{-1}$ and that the expectation value (9.6) of the Wilson loop may has a well-defined large N limit. In particular, in the zero coupling limit, the expectation value reduces to one.

9.1 Large N expansion and dimensional reduction

It is known [32] that only the planar diagrams contribute to the expectation value

$$
\langle \mathcal{A}_{\mu_1}^{a_1b_1}(x_1)\mathcal{A}_{\mu_2}^{a_2b_2}(x_2)\cdots\mathcal{A}_{\mu_n}^{a_nb_n}(x_n)\rangle_{YM_4} \tag{9.7}
$$

in the leading order of the large N expansion. See Fig. 9. However, it is extremely difficult to sum up the infinite number of terms belonging to the leading order of the large N expansion and to get the closed expression in the four-dimensional case. However, this does not exclude the possibility that the closed expression obtained by summing up all the leading diagram may exhibit the area law. In fact, this strategy has been carried out in the two-dimensional case and successfully has lead to the area law, see e.g., [60].

For the planar Wilson loop C, we have already shown that the Parisi-Sourlas dimensional reduction occurs and that the TQFT sector reduces to the two-dimensional coset NLS model, i.e. the NLS model on the flag space F_{N-1}. Hence, we obtain

$$
\left\langle e^{i\oint_C\omega}\right\rangle_{TQFT_4} = \left\langle e^{i\oint_C\omega}\right\rangle_{NLSM_2}, \tag{9.8}
$$

and

$$
\left\langle P_{a_1b_1}(x_1)\cdots P_{a_nb_n}(x_n)e^{i\oint_C\omega}\right\rangle_{TQFT_4} = \left\langle P_{a_1b_1}(x_1)\cdots P_{a_nb_n}(x_n)e^{i\oint_C\omega}\right\rangle_{NLSM_2}, \tag{9.9}
$$

where $x_1,\cdots,x_n \in C \subset \mathbf{R}^2$. For the quark in the fundamental representation of $SU(N)$, the relevant NLS model can be restricted to the CP^{N-1} model.

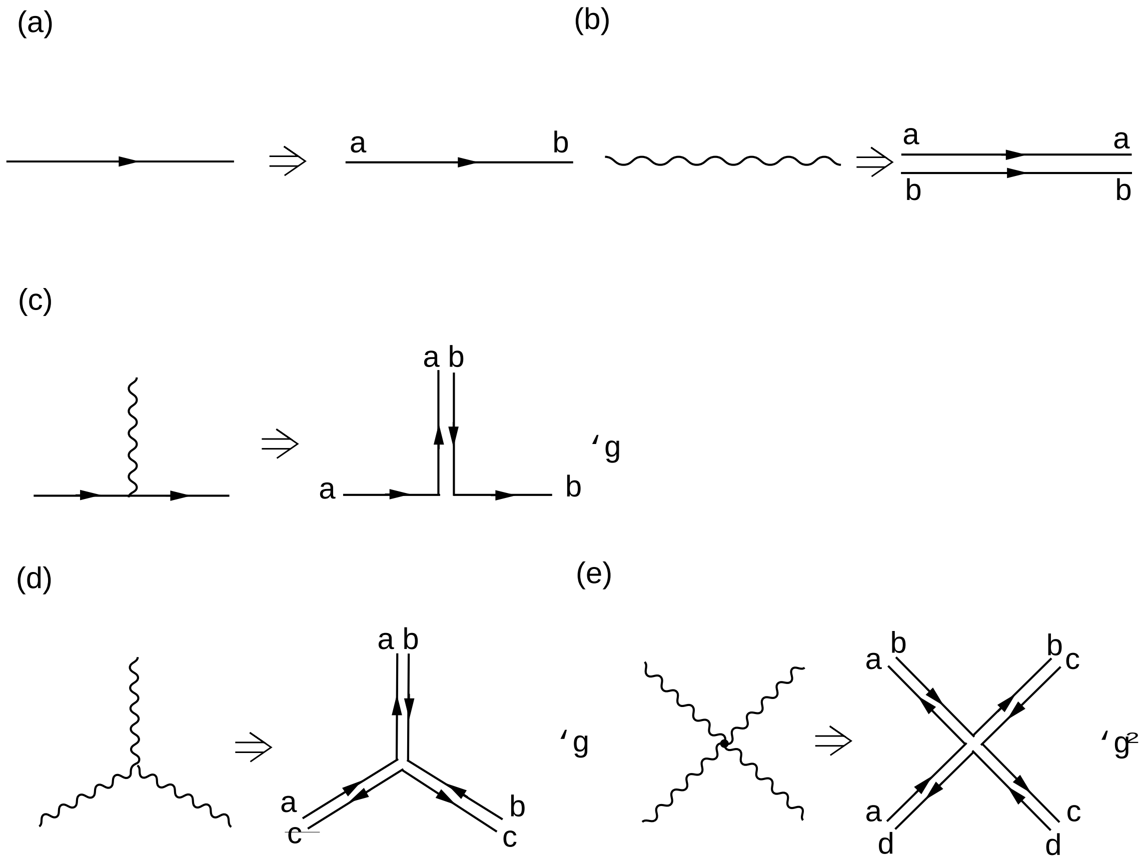


Figure 8: Feynmann rule and the corresponding large N rule (double line notation due to 't Hooft) in QCD. Propagators: (a) quark propagator, (b) gluon propagator. Vertices: (c) quark-gluon vertex (g_{YM}), (d) three-gluon vertex (g_{YM}), (e) four-gluon vertex (g^2_{YM}).

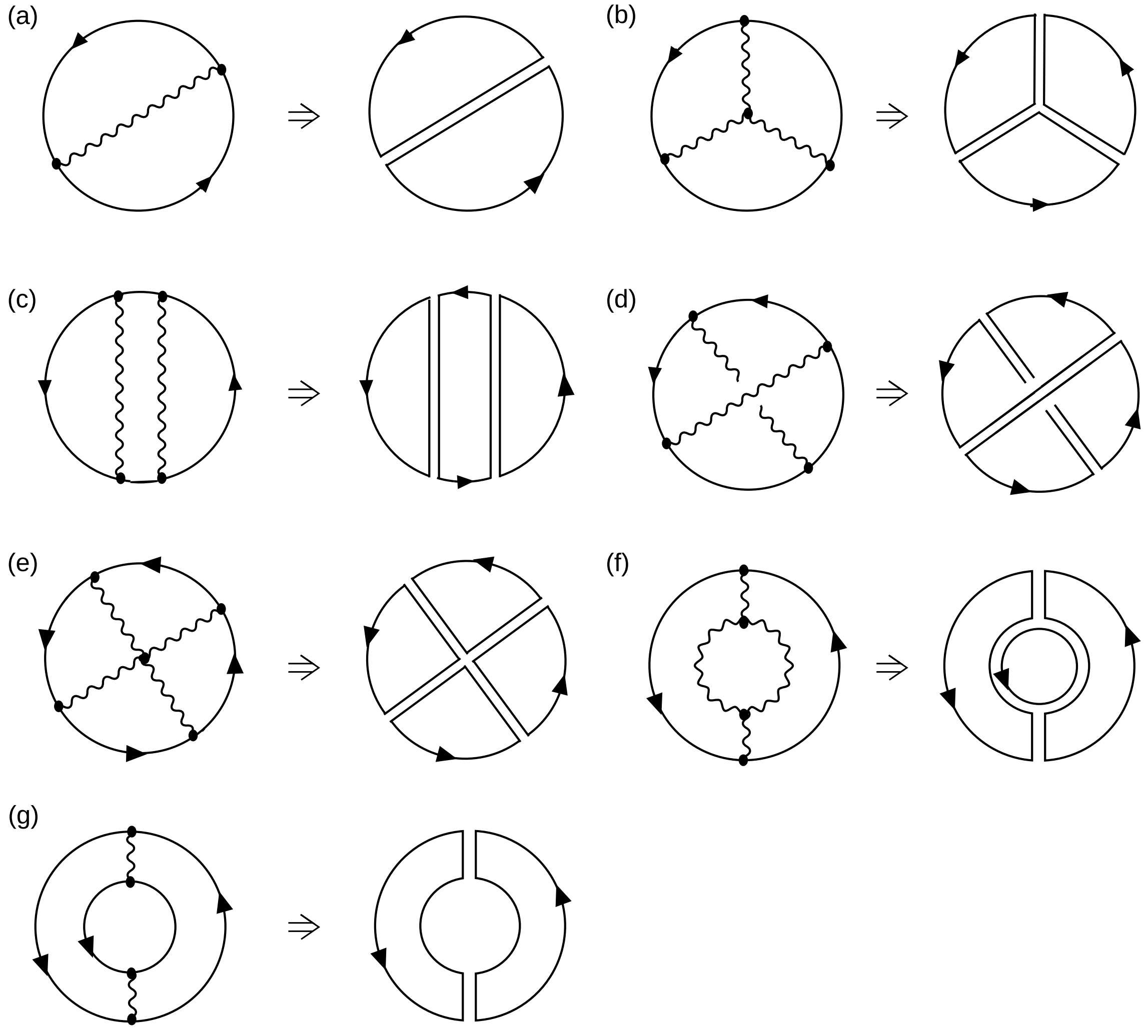


Figure 9: Examples of Feynmann diagrams and the corresponding double line notations which appear in calculating the expectation value of Wilson loop operator. The order of each diagram is estimated based on the rule given in Figure 8 as (a) $g_{YM}^2 N^2 = \lambda N$, (b) $g_{YM}^4 N^3 = \lambda^2 N$, (c) $g_{YM}^4 N^3 = \lambda^2 N$, (d) $g_{YM}^4 N = \lambda^2/N$, (e) $g_{YM}^6 N^4 = \lambda^3 N$, (f) $g_{YM}^4 N^3 = \lambda^2 N$, and (g) $g_{YM}^4 N^2 = \lambda^2 N^0$. [Here note that each contribution should be divided by the normalization factor N for the fundamental quark in agreement with the definition (9.6).] In the leading order of large N expansion, the leading contributions come from the planar diagrams, e.g., (a), (b), (c), (e), (f), which are furthermore classified by the order of λ. Note that the contribution from a nonplanar diagram (d) is suppressed in the large N. The diagram (g) is the vacuum polarization diagram due to quark–anti-quark pair creation and annihilation which is neglected in the pure Yang-Mills theory without dynamical quarks.

We come back to the expression (9.5) obtained by way of the NAST and apply the large N expansion to the (perturbative) deformation sector and the TQFT sector simultaneously. Taking the logarithm of the Wilson loop, therefore, we obtain

$$\begin{aligned}
&\ln\langle W^C[\mathcal{A}]\rangle_{YM_4} \\
= \ &\ln\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{CP_2^{N-1}} + \ln\left[1 - f[C] + O(\lambda^2/N^2)\right] \\
= \ &\ln\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{CP_2^{N-1}} - f[C] + O(\lambda^2/N^2), \qquad (9.10)
\end{aligned}$$

where we have defined the two-point correlation function,

$$G_{\mu\nu}^{ab,cd}(x,y) := \langle \mathcal{V}_\mu^{ab}(x)\mathcal{V}_\nu^{cd}(y)\rangle_{pYM_4}, \qquad (9.11)$$

and

$$f[C] := \frac{g^2}{2}\oint_C dx^\mu \oint_C dy^\nu G_{\mu\nu}^{ab,cd}(x,y)\frac{\langle P_{ab}(x)P_{cd}(y)\exp\left[i\oint_C \omega\right]\rangle_{CP_2^{N-1}}}{\langle \exp\left[i\oint_C \omega\right]\rangle_{CP_2^{N-1}}}. \qquad (9.12)$$

It should be remarked that the $f[C]$ is of order $O(\lambda/N)$. This is different from the result of the usual large N expansion of the Yang-Mills theory, i.e., diagram (a) of Fig. 9 which is of order $O(\lambda)$. This fact is shown as follows. In the previous article [17], it was shown that the perturbative sector obeys the Lorentz type gauge fixing, $\partial_\mu \mathcal{V}_\mu = 0$, by virtue of the background gauge. Here we adopt the Feynman gauge to simplify the calculation. Then the propagator for $\mathcal{V}_\mu^A$ reads

$$\langle \mathcal{V}_\mu^A(x)\mathcal{V}_\nu^B(y)\rangle_{pYM_4} = \delta^{AB}\delta_{\mu\nu}D(x,y), \quad D(x,y) := \frac{1}{4\pi^2}\frac{1}{|x-y|^2}. \qquad (9.13)$$

Then we find

$$\begin{aligned}
G_{\mu\nu}^{ab,cd}(x,y) &:= \langle \mathcal{V}_\mu^{ab}(x)\mathcal{V}_\nu^{cd}(y)\rangle_{pYM_4} &(9.14)\\
&= \langle \mathcal{V}_\mu^A(x)\mathcal{V}_\nu^B(y)\rangle_{pYM_4}(T^A)_{ab}(T^B)_{cd} &(9.15)\\
&= \delta_{\mu\nu}D(x,y)\sum_{A=1}^{N^2-1}(T^A)_{ab}(T^A)_{cd}, &(9.16)
\end{aligned}$$

where for $G = SU(N)$

$$\sum_{A=1}^{N^2-1}(T^A)_{ab}(T^A)_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right). \qquad (9.17)$$

This implies that

$$\sum_{A=1}^{N^2-1}(T^AT^A)_{ab} = \frac{N^2-1}{2N}\delta_{ab} = C_2(R)\delta_{ab}, \qquad (9.18)$$

where $C_2(R)$ is the quadratic (second order) Casimir invariant of the fundamental representation. If $G = U(N)$, the relation is simplified as $\sum_{A=1}^{N^2}(T^A)_{ab}(T^A)_{cd} = \frac{1}{2}\delta_{ad}\delta_{bc}$. The difference between $SU(N)$ and $U(N)$ disappears in the large N limit. In the leading order, we can put (see Appendix E)

$$\frac{\langle P_{ab}(x)P_{cd}(y)\exp\left[i\oint_C \omega\right]\rangle_{CP_2^{N-1}}}{\langle \exp\left[i\oint_C \omega\right]\rangle_{CP_2^{N-1}}} \quad \cong \quad \langle P_{ab}(x)P_{cd}(y)\rangle_{CP_2^{N-1}}. \tag{9.19}$$

Thanks to the $SU(N)$ invariance, it is easy to see that

$$\langle P_{ab}(x)P_{cd}(y)\rangle_{CP_2^{N-1}} = \left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right) Q(x,y), \tag{9.20}$$

where

$$Q(x,y) = \frac{\langle \bar{\phi}_a(x)\phi_b(x)\bar{\phi}_b(y)\phi_a(y)\rangle_{CP_2^{N-1}} - \frac{1}{N}\langle \bar{\phi}_a(x)\phi_a(x)\bar{\phi}_b(y)\phi_b(y)\rangle_{CP_2^{N-1}}}{N^2-1}. \tag{9.21}$$

This leads to

$$\begin{aligned}
& g^2 G_{\mu\nu}^{ab,cd}(x,y)\frac{\langle P_{ab}(x)P_{cd}(y)\exp\left[i\oint_C \omega\right]\rangle_{CP_2^{N-1}}}{\langle \exp\left[i\oint_C \omega\right]\rangle_{CP_2^{N-1}}} \\
= \quad & \delta_{\mu\nu}D(x,y)g^2\sum_{A=1}^{N^2-1}(T^A)_{ab}(T^A)_{cd}\left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right) Q(x,y) \\
= \quad & \delta_{\mu\nu}D(x,y)g^2\sum_{A=1}^{N^2-1}\left[\mathrm{tr}(T^AT^A) - \frac{1}{N}\mathrm{tr}(T^A)\mathrm{tr}(T^A)\right] Q(x,y) \\
= \quad & \delta_{\mu\nu}D(x,y)g^2\frac{N^2-1}{2}Q(x,y).
\end{aligned} \tag{9.22}$$

Note that $Q(x,x)$ is of order $O(N^{-2})$ and hence the $f[C]$ is of order $O(\lambda/N)$. This is because we consider the large N expansion of the CP^{N-1} model (see Appendix D) with the Lagrangian (see(C.23)),

$$\mathcal{L}_{CP^{N-1}} = \frac{N}{g_0^2}|(\partial_\mu + iV_\mu(x))\phi^\alpha(x)|^2, \quad g_0^2 := \frac{g_{YM}^2 N}{4\pi}, \tag{9.23}$$

with the composite gauge field,

$$V_\mu(x) = \frac{i}{2}(\bar{\phi}^\alpha(x)\partial_\mu\phi^\alpha(x) - \partial_\mu\bar{\phi}^\alpha(x)\phi^\alpha(x)), \tag{9.24}$$

under the constraint,

$$\phi^\dagger(x)\phi(x) := \bar{\phi}^a(x)\phi^a(x) = 1. \tag{9.25}$$

It is not difficult to show that the above estimation gives the correct order for the higher-order terms, e.g., (b), (e) in Fig. 9, by making use of the relations (4.73) and (4.74). For example, $g^3\langle\mathcal{V}^A_{\mu_1}(x_1)\mathcal{V}^B_{\mu_2}(x_2)\mathcal{V}^C_{\mu_n}(x_n)\rangle_{pYM_4}$ is proportional to $ig^4 f^{ABC}$, and $ig^4 f^{ABC}\mathrm{tr}[T^AT^BT^C]/N^3 = -g^4 f^{ABC}f^{ABC}/(4N^3) = O(g^4)$, since $f^{ABC}f^{ABD} = C_2(Adj)\delta^{CD}$ and $C_2(Adj) = N$.

Another way to understand this result is based on the idea of the reduction of degrees of freedom which are responsible to the Wilson loop. The flag space has dimension, $\mathrm{dim}F_{N-1} = N(N-1)$, whereas the CP^{N-1} has the dimension, $\mathrm{dim}CP^{N-1} = 2(N-1)$. Therefore, the relevant degrees of freedom is reduced for the fundamental quark in the large N, since $\mathrm{dim}CP^{N-1} \cong 2\mathrm{dim}F_{N-1}/N$ for large N. Indeed, this result is expected from the NAST given by (4.54), i.e.,

$$W^C[\mathcal{A}] = \int [d\mu(\xi)]_C \exp\left(ig\oint_C a\right). \tag{9.26}$$

The Abelian gauge field $a = \mathrm{tr}(\mathcal{H}\mathcal{A})$ has only two physical degrees of freedom, while the non-Abelian gauge field $\mathcal{A} = \mathcal{A}^AT^A$ in the Wilson loop (9.6) has $2(N^2-1)$ components. Thus the large N expansion is reduced to the perturbative expansion in the coupling constant g. In this sense, the large N expansion combined with the NAST justifies the identification of the deformation part with the perturbative one.

Then we find that the $f[C]$ is of order $O(\lambda/N)$. Therefore, in the leading order of the large N expansion, the static potential and the string tension are given by

$$V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{CP_2^{N-1}} + \lim_{T\to\infty}\frac{f[C]}{T} + O(\lambda^2/N^2), \tag{9.27}$$

$$\sigma = -\lim_{R,T\to\infty}\frac{1}{RT}\ln\left\langle \exp\left[i\oint_C \omega\right]\right\rangle_{CP_2^{N-1}} + \lim_{R,T\to\infty}\frac{f[C]}{RT} + O(\lambda^2/N^2). \tag{9.28}$$

Now we proceed to estimate the second term, $f[C]$. We will show that the second term gives at most the perimeter law so that the area law (if any) is provided by the first term. Because of the factor $\delta_{\mu\nu}$ in $\langle\mathcal{V}^A_\mu(x)\mathcal{V}^B_\nu(y)\rangle_{pYM_4}$, only integration between parallel sides dx and dy gives a contribution to $f[C]$. Thus $f[C]$ is reduced to

$$f[C] = \frac{g^2}{4}\oint_C dx^\mu \oint_C dy^\mu D(x,y)G_C(x,y), \tag{9.29}$$

where we have defined the correlation function for the composite operators,

$$\begin{aligned} G_C(x,y) &:= 2(T^A)_{ab}(T^A)_{cd}\frac{\langle P_{ab}(x)P_{cd}(y)\exp\left[i\oint_C\omega\right]\rangle_{CP_2^{N-1}}}{\langle\exp\left[i\oint_C\omega\right]\rangle_{CP_2^{N-1}}} && (9.30)\\ &\cong 2(T^A)_{ab}(T^A)_{cd}\left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right)Q(x,y) && (9.31)\\ &= (N^2-1)Q(x,y). && (9.32)\end{aligned}$$

First, if we restrict our consideration to the topologically trivial configurations, i.e.,

$$n^A(x)n^A(y) \equiv P_{ab}(x)P_{cd}(y)(T^A)_{ab}(T^A)_{cd} \cong n^A(\infty)n^A(\infty) \equiv \frac{1}{2}(1-N^{-1}), \tag{9.33}$$

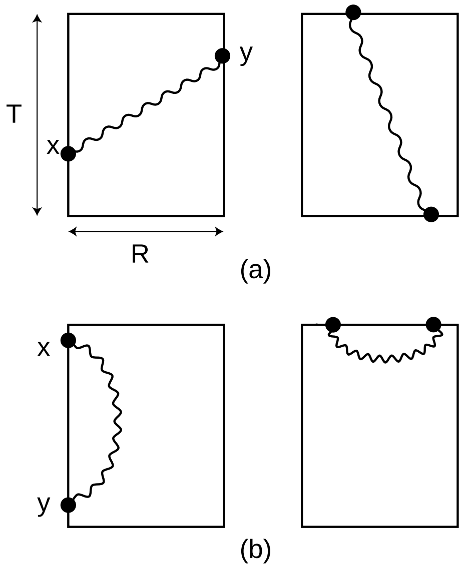


Figure 10: A rectangular Wilson loop and contribution to the Wilson integral in which x and y run over (a) opposite sides, and (b) same sides.

then we obtain $G_C(x,y) \cong 1 + O(N^{-1})$ and

$$f[C] \cong \frac{\lambda}{2N}(1+O(N^{-1}))h[C], \quad h[C] := \frac{1}{2}\oint_C dx^\mu \oint_C dy^\mu D(x,y). \tag{9.34}$$

By taking into account all the contributions from parallel sides dx and dy (see Fig.10), $h[C]$ is calculated as, for $T > R \gg 1$,

$$h[C] = -\frac{1}{4\pi}\frac{T}{R} + \frac{1}{2\pi^2}\frac{T+R}{\epsilon} + \frac{1}{2\pi^2}\ln\frac{R}{\epsilon}, \tag{9.35}$$

where ϵ is the ultraviolet cutoff to avoid the coincidence of x and y, see Appendix of [15]. In $h[C]$, the first term corresponds to the Coulomb potential in four dimensions,

$$V_C(R) = -\frac{g^2}{4\pi}\frac{1}{R} + \text{const.} + O(\lambda^2/N^2), \tag{9.36}$$

and the second term in $h[C]$ corresponds to the self-energy of quark and anti-quark. Furthermore, if we take into account the order $O(g^4)$ correction, the coupling constant begins to run and the bare coupling g in (9.36) is replaced by the running coupling constant $g = g(\mu)$, see e.g. Kogut [61].[17] In the topologically trivial case, therefore, the second term $f[C]$ can not give non-vanishing string tension.

Next, we consider the topologically non-trivial case. We begin to estimate (9.29) for the circular Wilson loop C with a diameter R, see Fig.11. If we avoid the coinciding case $x = y$, $f[C]$ has a contribution only when x and y are opposite ends of a diameter,

[17]The contribution up to $O(\lambda^2)$ in the leading order diagrams (planar diagram) in the large N expansion leads to the different running coupling from that in the usual perturbative calculation in the coupling constant g.

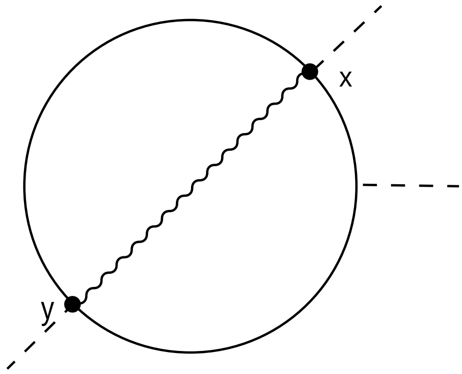


Figure 11: A circular Wilson loop.

i.e., $|x-y| = R$. Therefore, $D(x,y)$ and $G(x,y)$ are functions of R due to translational invariance. For any $x, y \in C$, $|x-y| = R$ and

$$\oint_C dx^\mu \oint_C dy^\mu D(R) = \frac{\pi^2 R^2}{4\pi^2 R^2} = \frac{1}{4}. \tag{9.37}$$

Hence we obtain

$$f[C] = \frac{\lambda}{8N} G_C(x,y). \tag{9.38}$$

It is almost clear that the $f[C]$ is not sufficient to give non-vanishing string tension, since $G_C(x,y)$ exhibits exponential decay for large $R := |x-y|$. Note that $P_{ab}(x)$ and $\oint_C \omega = \int_S d\omega$ are U(1) gauge invariant quantities, so that $G_C(x,y)$ is also U(1) gauge invariant. In the large N expansion, we can give more precise estimate of the second term, see Appendix E.

Finally, we consider the topologically nontrivial case for a rectangular Wilson loop C with side lengths R and T, see Fig.10. In this case, we can not give precise estimate of the second term, since we can not perform the integration exactly. In the leading order of $1/N$ expansion, it turns out that

$$G_C(x,y) \cong \tilde{G}(x,y) := 2(T^A)_{ab}(T^A)_{cd}\langle P_{ab}(x)P_{cd}(y)\rangle_{CP_2^{N-1}}, \tag{9.39}$$

and that $\tilde{G}(x,y)$ decays exponentially for sufficiently large $|x-y|$, see Appendix E. Note that x and y are located on the opposite sides of the rectangular Wilson loop. Therefore, there exists an uniform upper bound, i.e.,

$$|\tilde{G}(x,y)| \leq \tilde{G}(R). \tag{9.40}$$

Hence, there exists an upper bound for $f[C]$ for sufficiently large Wilson loop such that $T > R \gg 1$,

$$|f[C]| \leq \frac{\lambda}{2N}\tilde{G}(R)h[C], \tag{9.41}$$

where $h[C]$ is calculated in the same way as (9.35) by taking into account all the contributions from parallel sides dx and dy (see Fig.10). Therefore the second term $f[C]$ can not give non-vanishing string tension in the topologically nontrivial case.

Thus, within this reformulation, the area law of the non-Abelian Wilson loop and the linear static potential in four-dimensional $SU(N)$ Yang-Mills theory is realized if and only if the diagonal Wilson loop $\left\langle \exp\left(i\oint_C \omega\right) \right\rangle_{CP_2^{N-1}}$ in two-dimensional NLS model exhibits the area law,

$$\left\langle \exp\left(i\oint_C \omega\right) \right\rangle_{CP_2^{N-1}} \equiv \left\langle \exp\left(i\int_S \Omega_K\right) \right\rangle_{CP_2^{N-1}} \sim \exp(-\sigma_0 TR). \tag{9.42}$$

In other words, the area law or the linear potential between the fundamental quark and anti-quark is obtained from the topological $TQFT_4$ piece alone, i.e.,

$$\left\langle \exp\left(i\oint_C \omega\right) \right\rangle_{TQFT_4} = \left\langle \exp\left(i\oint_C \omega\right) \right\rangle_{CP_2^{N-1}} \sim \exp(-\sigma_0 TR). \tag{9.43}$$

Anyway, the derivation of the area law is reduced to the two-dimensional problem.

It should be remarked that only the total static potential,

$$V(R) = \sigma_0 R - \frac{g^2(\mu)}{4\pi}\frac{1}{R} + \text{const.}, \tag{9.44}$$

is gauge invariant, so the linear potential piece alone is not gauge invariant. However, in the large R limit $R \to \infty$, the linear potential is dominant in $V(R)$ so that the linear potential piece becomes substantially gauge invariant.

9.2 Area law in the leading order of the large N expansion

By the rescaling of the field ϕ in the Lagrangian (9.23), another form of the Lagrangian of the CP^{N-1} model is obtained as

$$\mathcal{L}_{CP^{N-1}} = \partial_\mu \bar{\phi}^\alpha(x)\partial_\mu \phi^\alpha(x) + \frac{g_0^2}{4N}(\bar{\phi}^\alpha(x)\partial_\mu \phi^\alpha(x) - \partial_\mu \bar{\phi}^\alpha(x)\phi^\alpha(x))^2, \tag{9.45}$$

with the constraint,

$$\phi^\dagger(x)\phi(x) := \bar{\phi}^a(x)\phi^a(x) = \frac{N}{g_0^2}, \quad g_0^2 := \frac{g_{YM}^2 N}{4\pi}. \tag{9.46}$$

It is useful to consider the Schwinger parameterization [62],

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} = e^{i\varphi}\left(1 + \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u}\right)^{-1} \begin{pmatrix} \mathbf{u} \\ \sqrt{\frac{N}{g_0^2}}(1 - \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u}) \end{pmatrix}, \tag{9.47}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}, \quad \mathbf{u}^\dagger \mathbf{u} := \bar{u}_\alpha u_\alpha \quad (\alpha = 1, \cdots, N-1). \tag{9.48}$$

Note that there is no constraint for the variable $\mathbf{u}$, since the Schwinger parameterization automatically satisfies the constraint (9.46). We can rewrite various quantities in terms of $\mathbf{u}$ without the constraint, e.g.,

$$\phi^\dagger d\phi - d\phi^\dagger \phi = \frac{\mathbf{u}^\dagger d\mathbf{u} - d\mathbf{u}^\dagger \mathbf{u}}{(1 + \frac{g^2}{4N}\mathbf{u}^\dagger \mathbf{u})^2} + 2i\frac{N}{g_0^2} d\varphi. \tag{9.49}$$

Then the $U(1)$ gauge field $V = V_\mu dx^\mu$ in the CP^{N-1} model is written as

$$V := \frac{i}{2}\frac{g_0^2}{N}(\phi^\dagger d\phi - d\phi^\dagger \phi) = \frac{i}{2}\frac{g_0^2}{N}\frac{\mathbf{u}^\dagger d\mathbf{u} - d\mathbf{u}^\dagger \mathbf{u}}{(1 + \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u})^2} - d\varphi. \tag{9.50}$$

We identify the complex coordinate w in the Kähler manifold with the Schwinger variable as

$$w_\alpha = \sqrt{\frac{g_0^2}{N}} \frac{u_\alpha}{1 - \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u}} \quad (\alpha = 1, \cdots, N-1). \tag{9.51}$$

This leads to

$$\bar{w}_\alpha dw_\alpha - d\bar{w}_\alpha w_\alpha = \frac{1}{(1 - \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u})^2}\frac{g_0^2}{N}(\bar{u}_\alpha du_\alpha - d\bar{u}_\alpha u_\alpha), \tag{9.52}$$

and

$$1 + \bar{w}_\alpha w_\alpha = \frac{(1 + \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u})^2}{(1 - \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u})^2}. \tag{9.53}$$

Then we find the expression of ω in terms of u,

$$\omega := \frac{i}{2}\frac{\bar{w}_\alpha dw_\alpha - d\bar{w}_\alpha w_\alpha}{1 + \bar{w}_\alpha w_\alpha} = \frac{i}{2}\frac{g_0^2}{N}\frac{\mathbf{u}^\dagger d\mathbf{u} - d\mathbf{u}^\dagger \mathbf{u}}{(1 + \frac{g_0^2}{4N}\mathbf{u}^\dagger \mathbf{u})^2}. \tag{9.54}$$

Thus the connection one-form is given by

$$V = \frac{i}{2}\frac{\bar{w}_\alpha dw_\alpha - d\bar{w}_\alpha w_\alpha}{1 + \bar{w}_\alpha w_\alpha} - d\varphi = \omega - d\varphi, \tag{9.55}$$

and the Abelian curvature two-form is equal to the Kähler two-form,

$$dV = d\omega = \Omega_K = i g_{\alpha\bar{\beta}} dw_\alpha \wedge d\bar{w}_\beta, \tag{9.56}$$

$$g_{\alpha\bar{\beta}} = \frac{\Delta \delta_{\alpha\beta} - \bar{w}_\alpha w_\beta}{\Delta}, \quad \Delta := 1 + |||w|||^2. \tag{9.57}$$

By way of the variable u, we have found that the connection one-form ω appearing in the NAST is equal to the gauge-invariant part of V. Therefore the diagonal Wilson loop for CP^{N-1} model is equal to

$$\left\langle \exp\left(i \oint_C \omega\right) \right\rangle_{CP^{N-1}} = \left\langle \exp\left(i \oint_C V\right) \right\rangle_{CP^{N-1}}. \tag{9.58}$$

For CP^1 model, this reduces (6.51) in the previous article [13] for $G = SU(2)$.

The expectation value $\left\langle \exp\left(i \oint_C V\right) \right\rangle_{CP^{N-1}}$ is calculated in the large N expansion based on the pioneering works [55, 56, 57] as shown in Appendix D. The result agrees with the result of Campostrini and Rossi [59]. In the leading order of $1/N$ expansion, the Wilson loop exhibits the area law for all non-self-intersecting loops,

$$\left\langle \exp\left(i \oint_C V\right) \right\rangle_{CP^{N-1}} = \exp\left[-\frac{6\pi}{N} m_\phi^2 |\mathrm{Area}(C)|\right], \tag{9.59}$$

where

$$m_\phi^2 = \mu^2 \exp\left[-\frac{16\pi^2}{N g_{YM}^2(\mu)}\right]. \tag{9.60}$$

Thus the string tension is obtained as

$$\sigma_0 = \frac{6\pi}{N} m_\phi^2. \tag{9.61}$$

Here m_ϕ is the mass of the CP^{N-1} field ϕ. The m_ϕ^2 is equal to the vacuum expectation value $\langle \sigma(x) \rangle$ of the Lagrange multiplier field σ from the correspondence, $\sigma(x)\bar{\phi}(x)\phi(x) \to m_\phi^2 \bar{\phi}(x)\phi(x)$. In the propagator of the vector field V, a massless pole appears. Hence the auxiliary vector field V becomes a dynamical gauge field, giving rise to a linear confining potential between ϕ and $\bar{\phi}$. On the other hand, the Lagrange multiplier field σ for the constraint (9.46) becomes massive so that it does not contribute to the confining potential between ϕ and $\bar{\phi}$. Thus we completes a proof of quark confinement in four-dimensional $SU(N)$ Yang-Mills theory based on the Wilson criterion in the leading order of large N expansion *within* our reformulation of the Yang-Mills theory.

10 Conclusion and discussion

We have given a new version of non-Abelian Stokes theorem for $G = SU(N)$ for $N \geq 2$ which reduces to the previous result [15] for $SU(2)$. This version of non-Abelian Wilson loop is very helpful to see the role played by the magnetic monopole in the calculation of the expectation value of the non-Abelian Wilson loop. Combining this non-Abelian Stokes theorem with the Abelian-projected effective gauge theory for $SU(N)$, we have explained the Abelian dominance for the Wilson loop in $SU(N)$ Yang-Mills gauge theory. For $SU(N)$ with $N \geq 3$, we must distinguish the maximal stability group $\tilde{H}$ and the residual gauge group H which is taken to be the maximal torus group $H = U(1)^{N-1}$.

In order to show magnetic monopole dominance and area law of the Wilson loop, we have used a novel reformulation of the Yang-Mills theory which has been proposed by one of the authors [13]. This reformulation is based on the identification of the Yang-Mills theory with the perturbative deformation of a topological quantum field theory. This framework deals with the gauge action S_{YM} and the gauge-fixing action S_{GF} on equal footing. The gauge action is characterized from the viewpoint of geometry of connections. On the other hand, the gauge-fixing part is related to the topological invariant (Euler characteristic) determined by a global topology. Therefore, the gauge-fixing part can have a geometric meaning from a global viewpoint, which has not been emphasized in the text books of quantum field theory so far, see [17] for more details.

Our approach relies heavily on a specific gauge, the MA gauge. In spite of the absence of elementary scalar field in Yang-Mills theory, the MA gauge made the existence of magnetic monopole possible. The magnetic monopole is considered as the topological soliton composed of gauge degrees of freedom, providing the composite scalar field. At least in this gauge, the (Parisi-Sourlas) dimensional reduction occurs due to the supersymmetry hidden in the gauge fixing part in the MA gauge. By making use of the non-Abelian Stokes theorem within this reformulation of the Yang-Mills theory, the derivation of the area law of the non-Abelian Wilson loop in four-dimensional Yang-Mills theory has been reduced to the two-dimensional problem of calculating the expectation value of the Abelian Wilson loop in the coset G/H non-linear sigma model. This is the main result of this article.

Especially, in order to show confinement of the fundamental quark in the four-dimensional $SU(N)$ Yang-Mills theory in the MA gauge, we have only to consider the two-dimensional CP^{N-1} model. From the topological point of view, the Abelian Wilson loop is equivalent to the area integral (enclosed by the Wilson loop) of the instanton density in two dimensional NLS model. This implies that the calculating the magnetic monopole contribution to the Wilson loop in four-dimensional Yang-Mills theory was translated into that of instanton contribution in two-dimensional NLS model, when the Wilson loop is contained in the two-dimensional plane, i.e. the Wilson loop is planar. In addition, the two-dimensional instanton is considered as a subclass of the four-dimensional Yang-Mills instanton, see [13]. This suggest that the area law of the Wilson loop will be derived by taking into account the contributions of a restricted class of Yang-Mills instantons. A Monte Carlo simulation on a lattice will be efficient in order to confirm this dimensionally reduced picture of quark confinement [63].

Moreover, the naive instanton calculus (dilute gas approximation) in the coset NLS model leads to area law of the Wilson loop of the original four-dimensional Yang-Mills theory. The improvements of the dilute gas approximation is necessary to confirm the area law based on the above picture. However, it is rather hard, as tried more than twenty years ago [54]. An advantage of the extension of the above strategy to $SU(N)$ with arbitrary N is that the large N systematic expansion becomes available. The large N calculation of the Wilson loop has shown the area law of the Wilson loop. These results confirm the area law of the Wilson loop. It is known that the large N Yang-Mills theory is related to the string theory. The correspondence between the instanton calculus and the large N expansion and the related issue in the large N expansion in Yang-Mills theory will also be discussed in a subsequent article [42]. These consideration will shed more light on the confining string picture [64].

A Normalization of the coherent state

In this appendix, we derivate the normalization facter N of coherent state $|\xi, \Lambda\rangle$ which parametrizes $G/\tilde{H}$.

$$
\begin{aligned}
N &:= \langle\Lambda| \exp\left[\bar{\tau}_\alpha E_\alpha\right] \exp\left[\tau_\beta E_{-\beta}\right] |\Lambda\rangle \qquad (A.1)\\
&= \sum_{K=0}^{\infty}\sum_{L=0}^{\infty} \frac{1}{K!}\frac{1}{L!} \bar{\tau}_{\alpha^1}\ldots\bar{\tau}_{\alpha^K}\tau_{\beta^1}\ldots\tau_{\beta^L} \langle\Lambda| E_{\alpha^1}\ldots E_{\alpha^K} E_{-\beta^1}\ldots E_{-\beta^L} |\Lambda\rangle . \qquad (A.2)
\end{aligned}
$$

A.1 $SU(3)$ coherent state

We use the positive root $\alpha^{(i)}$ and Dynkin index $[m, n]$ in $SU(3)$ which are defined by Fig.3 and (3.13). This definition may be rewritten as

$$
\alpha^{(1)} \cdot \vec{\Lambda} = \frac{m}{2},\ \alpha^{(3)} \cdot \vec{\Lambda} = \frac{n}{2}. \qquad (A.3)
$$

For $[m, n] = [m, 0]$ (resp. $[0, n]$), α and β run over $1, 2$ (resp. $2, 3$). Then the corresponding coherent state parametrizes CP^2. From orthogonality of the states which span the representation space, the terms contributing to N in (A.1) must satisfy the condition,

$$
\alpha^1 + \ldots + \alpha^K - \beta^1 - \ldots - \beta^L = 0. \qquad (A.4)
$$

In $[m, 0]$ case, this condition implies that the number of $\alpha^{(1)}$ and $\alpha^{(2)}$ in K positive roots is equal to that of $\alpha^{(1)}$ and $\alpha^{(2)}$ in L nagative ones. Since $[E_1, E_2] = 0$, we have only to estimate the following terms,

$$
N_{k,l} := \langle\Lambda| \underbrace{E_1\ldots E_1}_{k} \underbrace{E_2\ldots E_2}_{l} \underbrace{E_{-2}\ldots E_{-2}}_{l} \underbrace{E_{-1}\ldots E_{-1}}_{k} |\Lambda\rangle . \qquad (A.5)
$$

We begin with the term,

$$
N_{0,l} = \langle\Lambda| \underbrace{E_2\ldots E_2}_{l} \underbrace{E_{-2}\ldots E_{-2}}_{l} |\Lambda\rangle = N_{0,l} \left\langle \Lambda - l\alpha^{(2)} | \Lambda - l\alpha^{(2)} \right\rangle , \qquad (A.6)
$$

where we have used in the last equality the fact that $E_{-2}\ldots E_{-2} |\Lambda\rangle$ has a weight $\Lambda - l\alpha^{(2)}$ and is proportional to the state $\left|\Lambda - l\alpha^{(2)}\right\rangle$ which is normalized, i.e.,

$$
\left\langle \Lambda - l\alpha^{(2)} \middle| \Lambda - l\alpha^{(2)} \right\rangle = 1. \qquad (A.7)
$$

Exchanging the rightest E_2 with E_{-2} and using

$$[E_2, E_{-2}] = \alpha^{(2)} \cdot H, \tag{A.8}$$

$$\alpha^{(2)} \cdot H \left|\Lambda - j\alpha^{(2)}\right\rangle = (\frac{m}{2} - j) \left|\Lambda - j\alpha^{(2)}\right\rangle (0 < j < m), \tag{A.9}$$

we obtain the recursion relation for $N_{0,l}$,

$$\begin{aligned} N_{0,l} &= \{\frac{m}{2} - (l-1)\}N_{0,l-1} + \langle\Lambda| \underbrace{E_2 \ldots E_2}_{l-1} E_{-2}E_2 \underbrace{E_{-2} \ldots E_{-2}}_{l-1} |\Lambda\rangle \\ &= (\frac{m}{2} - (l-1) + \frac{m}{2} - (l-2) + \cdots + \frac{m}{2})N_{0,l-1} \\ &= \frac{l(m-l+1)}{2} N_{0,l-1}, \end{aligned} \tag{A.10}$$

where we have used $E_2 |\Lambda\rangle = 0$. From this relation and $N_{0,0} = \langle\Lambda|\Lambda\rangle = 1$, we obtain

$$N_{0,l} = \frac{(l!)^2 \binom{m}{l}}{2^l}. \tag{A.11}$$

Similarly,

$$N_{k,0} = \frac{(k!)^2 \binom{m}{k}}{2^k}. \tag{A.12}$$

We turn to the general terms (A.5),

$$\begin{aligned} N_{k,l} &= N_{k,0} \left\langle\Lambda - k\alpha^{(1)}\right| \underbrace{E_2 \ldots E_2}_{l} \underbrace{E_{-2} \ldots E_{-2}}_{l} \left|\Lambda - k\alpha^{(1)}\right\rangle \\ &= N_{k,0} \frac{k(m-k-l+1)}{2} \left\langle\Lambda - k\alpha^{(1)}\right| \underbrace{E_2 \ldots E_2}_{l-1} \underbrace{E_{-2} \ldots E_{-2}}_{l-1} \left|\Lambda - k\alpha^{(1)}\right\rangle \\ &= \frac{(k!)^2 \binom{m}{k}}{2^k} \frac{(l!)^2 \binom{m-k}{l}}{2^l}. \end{aligned} \tag{A.13}$$

Finally, the normalization facter N is given by

$$\begin{aligned} N &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(k+l)!^2} \binom{k+l}{l} \underbrace{\bar{\tau}_2 \ldots \bar{\tau}_2}_{l} \underbrace{\bar{\tau}_1 \ldots \bar{\tau}_1}_{k} \underbrace{\tau_2 \ldots \tau_2}_{l} \underbrace{\tau_1 \ldots \tau_1}_{k} N_{k,l} \\ &= \sum_{k=0}^{n} \sum_{l=0}^{n-k} (\frac{\bar{\tau}_2\tau_2}{2})^l (\frac{\bar{\tau}_1\tau_1}{2})^k \binom{m}{k} \binom{m-k}{l} \\ &= \left(1 + (\frac{\bar{\tau}_1\tau_1}{2}) + (\frac{\bar{\tau}_2\tau_2}{2})\right)^m \\ &= (1 + (\bar{w}_1 w_1) + (\bar{w}_2 w_2))^m = \exp K_{CP^2}(\bar{w}, w), \end{aligned} \tag{A.14}$$

where $w_1 := \tau_1/\sqrt{2}$ $w_2 := \tau_2/\sqrt{2}$ and $K_{CP^2}(w)$ is the Kähler potential of CP^2.

For F_2, i.e., $mn \neq 0$, α and β in (A.1) run over $1, 2, 3$. we apply almost the same step as in the case of CP^2. The result is

$$
\begin{aligned}
N &:= \langle\Lambda| \exp \sum_{\alpha=1}^{3} [\bar{\tau}_\alpha E_\alpha] \exp \sum_{\beta=1}^{3} [\tau_\beta E_{-\beta}] |\Lambda\rangle \\
&= \left(1 + |\frac{\tau_1}{\sqrt{2}}|^2 + |\frac{\tau_2}{\sqrt{2}} + \frac{\tau_1\tau_3}{4}|^2\right)^m \left(1 + |\frac{\tau_3}{\sqrt{2}}|^2 + |\frac{\tau_2}{\sqrt{2}} - \frac{\tau_1\tau_3}{4}|^2\right)^n \\
&= \left(1 + |w_1|^2 + |w_2|^2)\right)^m \left(1 + |w_3|^2 + |w_2 - w_1 w_3|^2\right)^n \\
&= \exp K_{F_2}(\bar{w}, w),
\end{aligned}
\tag{A.15}
$$

where $w_1 := \tau_1/\sqrt{2},\ w_2 := \tau_2/\sqrt{2} + \tau_1\tau_3/4,\ w_3 := \tau_3/\sqrt{2}$.

A.2 $SU(N)$ coherent state

In $SU(N)$, the simple root α^i and Dynkin index $[m_1, \cdots m_{N-1}]$ are defined by (3.47) and $\alpha^i \cdot \Lambda := m_i/2$. In general, the roots $E_{\pm\alpha}$ belonging to the coset group $G/\tilde{H}$ are nonorthogonal to the highest weight Λ[18]. We will restrict the case of $[m, 0, \cdots, 0]$, namely, only α^1 is nonorthogonal to Λ. Any positive root $\tilde{\alpha}$ is given by the linear combination of simple roots α^i. The positive roots $\tilde{\alpha}_i$ in (A.1) are nonorthogonal to Λ, so they must include α^1. It turns out that such roots are given by $\tilde{\alpha}_i := \alpha^1 + \cdots + \alpha^i$ $(i = 1, \cdots N-1)$, which are realized by $(E_{\tilde{\alpha}_i})_{kl} = \delta_{1,k}\delta_{i+1,l}/\sqrt{2}$ in $\mathbf{N}$ representation. The corresponding coherent state parametrizes CP^{N-1}. We can easily extend the CP^2 case to CP^{N-1} from the fact that $[E_{\tilde{\alpha}_i}, E_{\tilde{\alpha}_j}] = 0$ and $\tilde{\alpha}_i \cdot \vec{\Lambda} = m/2$. (This situation is same as $SU(3)$ case, i.e., $[E_1, E_2] = 0,\ \alpha^{(1)} \cdot \vec{\Lambda} = \alpha^{(2)} \cdot \vec{\Lambda} = m/2$.) Thus we obtain

$$
\begin{aligned}
N &:= \langle\Lambda| \exp \sum_{i=1}^{N-1} [\bar{\tau}_i E_{\tilde{\alpha}_i}] \exp \sum_{j=1}^{N-1} [\tau_j E_{-\tilde{\alpha}_j}] |\Lambda\rangle \\
&= \left(1 + \sum_{i=1}^{N-1} (\frac{\bar{\tau}_i\tau_i}{2})\right)^m =: \left(1 + \sum_{i=1}^{N-1} (\bar{w}_i w_i)\right)^m \\
&= \exp K_{CP^{N-1}}(\bar{w}, w).
\end{aligned}
\tag{A.16}
$$

B From CP^1 to CP^2

For an arbitrary element (ϕ_1, ϕ_2, ϕ_3) of W,

$$W = \mathbf{C}^3 - (0, 0, 0), \tag{B.1}$$

the whole set of ratios $\phi_1 : \phi_2 : \phi_3$ are called the complex projective plane and are denoted by $P^2(\mathbf{C})$ or $\mathbf{C}P^2$, namely,

$$(\alpha\phi_1 : \alpha\phi_2 : \alpha\phi_3) = (\phi_1 : \phi_2 : \phi_3), \quad \alpha \in \mathbf{C}, \quad \alpha \neq 0. \tag{B.2}$$

[18] From the difinition of coherent state (2.9), $E_{\pm\alpha} |\Lambda\rangle \neq 0,\ [E_\alpha, E_{-\alpha}] |\Lambda\rangle = \alpha \cdot \Lambda |\Lambda\rangle \neq 0$

Defining the subset $U_a (a = 1, 2, 3)$ of $\mathbf{C}P^2$,

$$U_a = \{(\phi_1 : \phi_2 : \phi_3) \in \mathbf{C}P^2; \phi_j \neq 0\} \subset \mathbf{C}P^2, \tag{B.3}$$

we observe that

$$(\phi_1 : \phi_2 : \phi_3) = (1 : \frac{\phi_2}{\phi_1} : \frac{\phi_3}{\phi_1}) \in U_1. \tag{B.4}$$

The mapping φ_1 from U_1 to $\mathbf{C}^2$,

$$\varphi_1 : (\phi_1 : \phi_2 : \phi_3) \in U_1 \to (\frac{\phi_2}{\phi_1} : \frac{\phi_3}{\phi_1}) \in \mathbf{C}^2, \tag{B.5}$$

is bijection, i.e., surjection (onto-mapping) and injection (one-to-one mapping). The inverse mapping is given by

$$\varphi_1^{-1} : (x, y) \in \mathbf{C}^2 \to (1 : x : y) \in U_1. \tag{B.6}$$

Similarly, the following maps φ_2, φ_3 are also bijections from U_2, U_3 to $\mathbf{C}^2$,

$$\varphi_2 : (\phi_1 : \phi_2 : \phi_3) \in U_2 \to (\frac{\phi_1}{\phi_2} : \frac{\phi_3}{\phi_2}) \in \mathbf{C}^2, \tag{B.7}$$

$$\varphi_3 : (\phi_1 : \phi_2 : \phi_3) \in U_3 \to (\frac{\phi_1}{\phi_3} : \frac{\phi_2}{\phi_3}) \in \mathbf{C}^2. \tag{B.8}$$

Since

$$\mathbf{C}P^2 = U_1 \cup \{(0 : \phi_2 : \phi_3)\}, \quad (\phi_2 : \phi_3) \neq 0, \tag{B.9}$$

$(\phi_2 : \phi_3)$ determines a point in $\mathbf{C}P^2$. Conversely, for a point $(b_1 : b_2)$ in $\mathbf{C}P^1$, $(0 : b_1 : b_2)$ defines a point in $\mathbf{C}P^2 - U_1$, i.e., the map

$$(0 : \phi_2 : \phi_3) \in \mathbf{C}P^2 - U_1 \to (\phi_2 : \phi_3) \in \mathbf{C}P^1 \tag{B.10}$$

is the one-to-one and onto mapping. Then we can identify $\mathbf{C}P^2 - U_1$ with $\mathbf{C}P^1$, $\mathbf{C}P^2 - U_1 \cong \mathbf{C}P^1$.

On the other hand, the identification of U_1 and $\mathbf{C}^2$, i.e., $U_1 \cong \mathbf{C}^2$ by the mapping φ_1 leads to

$$\mathbf{C}P^2 = U_1 \cup \mathbf{C}P^1 \cong \mathbf{C}^2 \cup \mathbf{C}P^1. \tag{B.11}$$

Using $\mathbf{C}P^1 = \mathbf{C}^1 \cup \{(0 : 1)\}$, we can write $\mathbf{C}P^2 = \mathbf{C}^2 \cup \mathbf{C}^1 \cup \{(0 : 1)\} \equiv \mathbf{C}^2 \cup \mathbf{C}^1 \cup \mathbf{C}^0$. $\mathbf{C}P^2 - U_1$ is called the line at infinity and is denoted by ℓ_∞, i.e., $\mathbf{C}P^2 - U_1 = \ell_\infty$. So we can also write

$$\mathbf{C}P^2 = U_1 \cup \ell_\infty, \quad U_1 \cong C^2, \quad \ell_\infty \cong \mathbf{C}P^1, \tag{B.12}$$

where $\mathbf{C}^2$ is called the complex affine plane. The homogeneous coordinate $(\phi_1 : \phi_2 : \phi_3)$ is related to an element (x, y) in the affine plane as

$$x = \frac{\phi_2}{\phi_1}, \quad y = \frac{\phi_3}{\phi_1}, \quad \ell_\infty \cong \{\phi_1 = 0\}. \tag{B.13}$$

$\mathbf{C}P^2$ is obtained as $\mathbf{C}P^2 = \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$ by gluing three affine planes $\mathcal{U}, \mathcal{V}, \mathcal{W}$ by the relation,

$$u_1 = \frac{1}{v_1}, u_2 = \frac{v_2}{v_1}, \quad v_1 = \frac{w_1}{w_2}, v_2 = \frac{1}{w_2}, \quad w_1 = \frac{1}{u_2}, w_2 = \frac{u_1}{u_2}, \tag{B.14}$$

where $(u_1, u_2), (v_1, v_2), (w_1, w_2)$ are coordinates of $\mathcal{U}, \mathcal{V}, \mathcal{W}$ via homogeneous coordinates of $\mathbf{C}P^2$,

$$(u_1, u_2) = (\frac{\phi_2}{\phi_1}, \frac{\phi_3}{\phi_1}), \quad (v_1, v_2) = (\frac{\phi_1}{\phi_2}, \frac{\phi_3}{\phi_2}), \quad (w_1, w_2) = (\frac{\phi_1}{\phi_3}, \frac{\phi_2}{\phi_3}). \tag{B.15}$$

Note that the way of gluing affine planes is not unique. In fact, $\mathbf{C}P^2$ is obtained from the following gluing,

$$u_1 = \frac{w_2}{w_1}, u_2 = \frac{1}{w_1}, \quad v_1 = \frac{1}{u_1}, v_2 = \frac{u_2}{u_1}, \quad w_1 = \frac{v_1}{v_2}, w_2 = \frac{1}{v_2}. \tag{B.16}$$

C Nonlinear sigma model of the flag space

C.1 Correspondence between $SU(N)/T$ and $SL(n, C)/B$

When an element ξ in F_{N-1} is expressed by the complex coordinate,

$$\xi = \begin{pmatrix} 1 & w_1 & w_2 & \cdots & \cdots & w_n \\ 0 & 1 & w_{n+1} & \cdots & \cdots & w_{2n-1} \\ 0 & 0 & 1 & w_{2n} & \cdots & w_{3n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & w_{n(n+1)/2} \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}^T \in F_n, \tag{C.1}$$

we find $\det \xi = 1$. Hence ξ is an element of $SL(N, C)$. There is an isomorphism, $SU(n)/T \cong SL(n, C)/B$. However, ξ in this form is not necessarily unitary. The corresponding unitary matrix $V \in SU(N)$ is obtained as follows. First, the ξ as an element of $SL(N, C)$ is expressed by the Column vectors,

$$\xi = (E_1, E_2, \cdots, E_N) \in SL(N, C) = SU(N)^C. \tag{C.2}$$

By applying the Gramm-Schmidt orthogonalization, we can obtain a set of mutually orthogonal vectors $(E'_1, E'_2, \cdots, E'_N)$ from $(E_1, E_2, \cdots, E_N)$,

$$\begin{aligned} E'_1 &:= E_1, \\ E'_2 &:= E_2 - \frac{(E_2, E'_1)}{(E'_1, E'_1)} E'_1, \\ &\vdots \\ E'_N &:= E_N - \frac{(E_N, E'_{N-1})}{(E'_{N-1}, E'_{N-1})} E'_{N-1} - \cdots - \frac{(E_N, E'_1)}{(E'_1, E'_1)} E'_1, \end{aligned} \tag{C.3}$$

where the inner product is defined by $(E_i, E_j) := E_i^T \cdot \bar{E}_j$. Using the normalized vectors $e_j := E'_j/||E'_j||$, we obtain the element in $SU(N)$,

$$V = (e_1, e_2, \cdots, e_N) \in SU(N). \tag{C.4}$$

In fact, $\det V = 1$ and V is unitary, since

$$V^\dagger V = \begin{pmatrix} \bar{e}_1^T \\ \vdots \\ \bar{e}_N^T \end{pmatrix} (e_1, e_2, \cdots, e_N), \quad (V^\dagger V)_{ij} = \bar{e}_i^T \cdot e_j = (e_i, e_j) = \delta_{ij}. \tag{C.5}$$

In general, the unitary matrix V is related to its complexfication V^C by

$$V = V^C B, \quad V \in SU(N), \quad V^C \in SL(N, C), \tag{C.6}$$

where B is an upper triangular matrix. This is nothing bu the Iwasawa decomposition. [19] Since the upper triangular matrices for a group,

$$V^C = VB^{-1} = VB', \tag{C.8}$$

where $B' = B^{-1}$ is also upper triangular. This implies $\xi = VB$ by the above construction. Therefore, V is indeed an element of $SU(N)$ corresponding to ξ. Note that the multiplication by the matrix B leaves the highest-weight state $|\Lambda\rangle$ invariant, so that

$$\xi|\Lambda\rangle = VB|\Lambda\rangle = V|\Lambda\rangle. \tag{C.9}$$

For example, $e_1 = E'_1/||E'_1|| = E_1/||E_1||$, i.e.,

$$e_1 = \frac{1}{\Delta^{1/2}} \begin{pmatrix} 1 \\ -w_1 \\ \vdots \\ -w_N \end{pmatrix}, \quad \Delta := 1 + ||w||^2 = 1 + \sum_{a=1}^{N} |w_a|^2. \tag{C.10}$$

The Mauer-Cartan form is

$$V^{-1}dV = V^\dagger dV = \begin{pmatrix} \bar{e}_1^T \\ \vdots \\ \bar{e}_N^T \end{pmatrix} (de_1, \cdots, de_N) = \begin{pmatrix} \bar{e}_1^T de_1 & \bar{e}_1^T de_2 & \cdots & \bar{e}_1^T de_N \\ \bar{e}_2^T de_1 & \bar{e}_2^T de_2 & \cdots & \bar{e}_2^T de_N \\ \cdots & \cdots & \cdots & \cdots \\ \bar{e}_N^T de_1 & \bar{e}_N^T de_2 & \cdots & \bar{e}_N^T de_N \end{pmatrix}, \tag{C.11}$$

which is decomposed into the diagonal and off-diagonal parts,

$$V^{-1}dV = \sum_{i=1}^{N} \bar{e}_i^T de_i I_{ii} + \sum_{a \neq b} \bar{e}_a^T de_b E_{ab}. \tag{C.12}$$

[19] Any element $g_c \in G^C$ may be factorized as

$$g_c = gb, \quad g \in G, \quad b \in B, \tag{C.7}$$

in a unique fashion, up to torus elements, which are common to G and B.

In the fundamental representation,

$$\omega = \langle \Lambda | i V^{-1} dV | \Lambda \rangle = i (V^{-1} dV)_{11} = i \bar{e}_1^T de_1, \tag{C.13}$$

$$n^A = \langle \Lambda | V^{-1} T^A V | \Lambda \rangle = \bar{e}_1^T (T^A) e_1. \tag{C.14}$$

Hence, substituting (C.10) into (C.13), we obtain

$$\omega = \frac{i}{2} \frac{\bar{w}_a dw_a - d\bar{w}_a w_a}{\Delta}. \tag{C.15}$$

The Lagrangian density of the coset G/H, i.e. flag NLS model,

$$\mathcal{L}_{NLSM} = \frac{\beta_g}{8} \mathrm{tr}_{G/H}(i V^{-1} \partial_\mu V i V^{-1} \partial_\mu V), \tag{C.16}$$

is written in terms of the off-diagonal elements as

$$\mathcal{L}_{NLSM} = \frac{\beta_g}{2} \sum_{a,b:a<b} (\Omega_\mu)_{ab} (\Omega_\mu)_{ab} = \frac{\beta_g}{2} \sum_{a,b:a<b} (e_a, \partial_\mu e_b)(e_a, \partial_\mu e_b), \tag{C.17}$$

where we have used $(e_i, e_j) := \bar{e}_i^T \cdot e_j = \delta_{ij}$.

Especially, the Lagrangian of CP^{N-1} model is obtained as a special case of (C.17) as follows. Using the definition (4.37), i.e.,

$$n^A = (U T^A U^\dagger)_{11} = U_{1a} (T^A)_{ab} \bar{U}_{1b} = U_{1a} (T^A)_{ab} U^\dagger_{b1}, \tag{C.18}$$

and $UU^\dagger = 1$, we find

$$\begin{aligned} \partial_\mu n^A \partial_\mu n^A &= \partial_\mu U_{1a} \partial_\mu U^\dagger_{a1} U_{1b} U^\dagger_{b1} + U_{1a} \partial U^\dagger_{a1} U_{1b} \partial_\mu U^\dagger_{b1} \\ &= \sum_{b=2}^{N} (i U \partial_\mu U^\dagger)_{1b} (i U \partial_\mu U^\dagger)_{1b} = \sum_{b=2}^{N} (e_1, \partial_\mu e_b)^2. \end{aligned} \tag{C.19}$$

Another expression is obtained if we use (4.36), i.e., $\phi_a = \bar{U}_{1a} = U^\dagger_{a1}$:

$$\partial_\mu n^A \partial_\mu n^A = \partial_\mu \bar{\phi} \cdot \partial_\mu \phi + (\bar{\phi} \cdot \partial_\mu \phi)(\bar{\phi} \cdot \partial_\mu \phi), \tag{C.20}$$

where we have used $\phi^\dagger \cdot \phi = 1$. Thus the Lagrangian of CP^{N-1} model is obtained as

$$\begin{aligned} \mathcal{L}_{CP^{N-1}} &= \frac{\beta_g}{2} \partial_\mu \mathbf{n} \cdot \partial_\mu \mathbf{n} \quad (\mu = 1, \cdots, d) && \text{(C.21)} \\ &= \frac{\beta_g}{2} \sum_{b=2}^{N} (e_1, \partial_\mu e_b)^2 && \text{(C.22)} \\ &= \frac{\beta_g}{2} g_{\alpha\bar{\beta}}(\phi) \frac{\partial \phi^\alpha}{\partial x_\mu} \frac{\partial \bar{\phi}^\beta}{\partial x_\mu}, && \text{(C.23)} \end{aligned}$$

where

$$g_{\alpha\bar{\beta}}(\phi) := \delta_{\alpha\beta} - \bar{\phi}_\alpha \phi_\beta. \tag{C.24}$$

This agrees with the Lagrangian obtained from the Kähler potential,

$$\mathcal{L}_{CP^{N-1}} = \frac{\beta_g}{2} g_{\alpha\bar{\beta}}(w)\partial_\mu w^\alpha \partial_\mu \bar{w}^\beta, \tag{C.25}$$

with

$$g_{\alpha\bar{\beta}}(w) = \frac{(1+|||w|||^2)\delta_{\alpha\beta} - \bar{w}_\alpha w_\beta}{(1+|||w|||^2)^2}. \tag{C.26}$$

The explicit construction of V is given as follows.

C.2 SU(2)

For $G = SU(2)$,

$$\xi = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} = (E_1, E_2), \quad E_1 = \begin{pmatrix} 1 \\ -w \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{C.27}$$

It is easy to see

$$(E_1, E_1) = 1 + w\bar{w}, \quad (E_2, E_1) = -\bar{w}. \tag{C.28}$$

Hence we obtain

$$e_1 = \Delta^{-1/2}\begin{pmatrix} 1 \\ -w \end{pmatrix}, \quad e_2 = \Delta^{-1/2}\begin{pmatrix} \bar{w} \\ 1 \end{pmatrix}, \quad \Delta := 1 + |w|^2, \tag{C.29}$$

and

$$V = (e_1, e_2) = \Delta^{-1/2}\begin{pmatrix} 1 & \bar{w} \\ -w & 1 \end{pmatrix}. \tag{C.30}$$

The elements of the one-form $V^{-1}dV$ is

$$e_1^T d\bar{e}_1 = \frac{1}{2}\Delta^{-1}(w d\bar{w} - \bar{w} dw), \tag{C.31}$$

$$e_2^T d\bar{e}_1 = -\Delta^{-1} d\bar{w}. \tag{C.32}$$

The Lagrangian of $F_1 = CP^1$ model reads

$$\mathcal{L}_{NLSM} = \frac{\beta_g}{2}\Delta^{-2}\partial_\mu w \partial_\mu \bar{w} = \frac{\beta_g}{2}\frac{1}{1+|w|^2}\partial_\mu w \partial_\mu \bar{w}. \tag{C.33}$$

C.3 SU(3)

For $G = SU(3)$,

$$\xi = \begin{pmatrix} 1 & 0 & 0 \\ -w_1 & 1 & 0 \\ -w_2 & w_3 & 1 \end{pmatrix}, \tag{C.34}$$

or,

$$E_1 = \begin{pmatrix} 1 \\ -w_1 \\ -w_2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \\ w_3 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{C.35}$$

Then we have

$$(E_1, E_1) = 1 + w_1\bar{w}_1 + w_2\bar{w}_2, \quad (E_2, E_1) = -\bar{w}_1 - w_3\bar{w}_2, \cdots. \tag{C.36}$$

The straightforward calculation leads to $V = (e_1, e_2, e_3) \in SU(3)$ with

$$\begin{aligned} e_1 &= (\Delta_1)^{-1/2} \begin{pmatrix} 1 \\ -w_1 \\ -w_2 \end{pmatrix}, \\ e_2 &= (\Delta_1\Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_1 + w_3\bar{w}_2 \\ 1 + |w_2|^2 - w_1\bar{w}_2 w_3 \\ -w_2\bar{w}_1 + w_3 + w_3|w_1|^2 \end{pmatrix}, \\ e_3 &= (\Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_2 - \bar{w}_1\bar{w}_3 \\ -w_3 \\ 1 \end{pmatrix}, \end{aligned} \tag{C.37}$$

where

$$\Delta_1 := 1 + |w_1|^2 + |w_2|^2, \quad \Delta_2 := 1 + |w_2 - w_1 w_3|^2 + |w_3|^2. \tag{C.38}$$

The off-diagonal elements of the one-form $V^{-1}dV$ are

$$\begin{aligned} e_2^T d\bar{e}_1 &= (\Delta_1)^{-1}(\Delta_2)^{-1/2}[(1 + |w_2|^2 - w_1\bar{w}_2 w_3)d\bar{w}_1 + (-w_2\bar{w}_1 + w_3 + w_3|w_1|^2)d\bar{w}_2], \\ e_3^T d\bar{e}_1 &= (\Delta_1)^{-1/2}(\Delta_2)^{-1/2}[d\bar{w}_2 - \bar{w}_3 d\bar{w}_1], \\ e_3^T d\bar{e}_2 &= (\Delta_1)^{-1/2}(\Delta_2)^{-1}[(w_1 + \bar{w}_3 w_2)(d\bar{w}_2 - \bar{w}_3 d\bar{w}_1) - \Delta_1 d\bar{w}_3]. \end{aligned} \tag{C.39}$$

For CP^2,

$$e_1 = (\Delta_1)^{-1/2} \begin{pmatrix} 1 \\ -w_1 \\ -w_2 \end{pmatrix}, \tag{C.40}$$

$$e_2 = (\Delta_1\Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_1 \\ 1 + |w_2|^2 \\ -w_2\bar{w}_1 \end{pmatrix}, \tag{C.41}$$

$$e_3 = (\Delta_2)^{-1/2} \begin{pmatrix} \bar{w}_2 \\ 0 \\ 1 \end{pmatrix}, \tag{C.42}$$

where

$$\Delta_1 := 1 + |w_1|^2 + |w_2|^2 = 1 + |||w|||^2, \quad \Delta_2 := 1 + |w_2|^2. \tag{C.43}$$

The off-diagonal elements of the one-form $V^{-1}dV$ are

$$\begin{aligned}
e_2^T d\bar{e}_1 &= (\Delta_1)^{-1}(\Delta_2)^{-1/2}[(1+|w_2|^2)d\bar{w}_1 - w_2\bar{w}_1 d\bar{w}_2], \\
e_3^T d\bar{e}_1 &= (\Delta_1)^{-1/2}(\Delta_2)^{-1/2}[d\bar{w}_2], \\
e_3^T d\bar{e}_2 &= (\Delta_1)^{-1/2}(\Delta_2)^{-1}[w_1 d\bar{w}_2]. \qquad \text{(C.44)}
\end{aligned}$$

The Lagrangian of CP^2 model is given by

$$\begin{aligned}
\mathcal{L}_{CP^2} &= \frac{\beta_g}{2}\sum_{b=2}^{3}|(e_1, \partial_\mu e_b)|^2 \qquad \text{(C.45)} \\
&= \frac{\beta_g}{2}\frac{1}{(1+|||w|||^2)^2}[(1+|w_2|^2)\partial_\mu w_1 \partial_\mu \bar{w}_1 + (1+|w_1|^2)\partial_\mu w_2 \partial_\mu \bar{w}_2 \\
&\qquad - \bar{w}_1 w_2 \partial_\mu w_1 \partial_\mu \bar{w}_2 - w_1 \bar{w}_2 \partial_\mu w_2 \partial_\mu \bar{w}_1]. \qquad \text{(C.46)}
\end{aligned}$$

D Large N expansion of CP^{N-1} model

The generating function of CP^{N-1} model is defined by

$$\begin{aligned}
Z[J, \bar{J}, J_\mu] &:= \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \prod_x \delta\left(|\phi(x)|^2 - \frac{N}{g_0^2}\right) \\
&\times \exp\left\{-S + \int d^2x[\bar{J}(x)\cdot\phi(x) + \bar{\phi}(x)\cdot J(x) + J_\mu(x)V_\mu(x)]\right\}, \qquad \text{(D.1)}
\end{aligned}$$

where S is the action of CP^{N-1} model,

$$\begin{aligned}
S &:= \int d^2x \left[\partial_\mu\bar{\phi}\cdot\partial_\mu\phi + \frac{g_0^2}{4N}(\bar{\phi}\cdot \overset{\leftrightarrow}{\partial_\mu} \phi)(\bar{\phi}\cdot \overset{\leftrightarrow}{\partial_\mu} \phi)\right] \qquad \text{(D.2)} \\
&= \int d^2x \left[\partial_\mu\bar{\phi}\cdot\partial_\mu\phi - \frac{N}{g_0^2}V_\mu V_\mu\right], \qquad \text{(D.3)}
\end{aligned}$$

and the auxiliary vector field V_μ is defined by

$$V_\mu(x) := \frac{g_0^2}{2N} i(\bar{\phi}(x)\cdot \overset{\leftrightarrow}{\partial_\mu} \phi(x)) = \frac{g_0^2}{2N} i(\bar{\phi}(x)\cdot\partial_\mu\phi(x) - \partial_\mu\bar{\phi}(x)\cdot\phi(x)). \qquad \text{(D.4)}$$

Introducing the Lagrange multiplier fields, $\sigma(x)$ and $A_\mu(x)$, we can rewrite

$$\begin{aligned}
&\prod_x \delta\left(|\phi(x)|^2 - \frac{N}{g_0^2}\right)\exp\left\{\int d^2x[\frac{N}{g_0^2}V_\mu(x)V_\mu(x) + J_\mu(x)V_\mu(x)]\right\}, \\
= &\int \mathcal{D}\sigma \int \mathcal{D}A_\mu \exp\Bigg[\int d^2x \Bigg\{\frac{i}{\sqrt{N}}\sigma\left(|\phi|^2 - \frac{N}{g_0^2}\right) - \frac{1}{N}A_\mu A_\mu|\phi|^2 - m^2|\phi|^2 \\
&+ \frac{1}{\sqrt{N}}A_\mu[i(\bar{\phi}\cdot \overset{\leftrightarrow}{\partial_\mu} \phi) + J_\mu] - \frac{g_0^2}{4N}J_\mu J_\mu\Bigg\}\Bigg], \qquad \text{(D.5)}
\end{aligned}$$

where we have inserted the mass term $m^2|\phi|^2$ for later convenience and have chosen the specific normalization for the field σ. [20] Then we obtain

$$
\begin{aligned}
Z[J,\bar{J},J_\mu] &:= \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\int\mathcal{D}\sigma\int\mathcal{D}A_\mu \exp\Big\{-\int d^2x\left[\bar{\phi}\cdot\Delta_B\phi+\frac{i\sqrt{N}}{g_0^2}\sigma\right] \\
&+ \int d^2x\left[\bar{J}\cdot\phi+\bar{\phi}\cdot J+\frac{1}{\sqrt{N}}A_\mu J_\mu-\frac{g_0^2}{4N}J_\mu J_\mu\right]\Big\}, \qquad \text{(D.6)}
\end{aligned}
$$

where

$$
\Delta_B := -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{N}}\sigma(x), \quad D_\mu := \partial_\mu + \frac{i}{\sqrt{N}}A_\mu(x). \qquad \text{(D.7)}
$$

This theory has global $SU(N)$ invariance corresponding to rotations of ϕ_a. Moreover, this theory has local $U(1)$ gauge invariance under the transformation,

$$
\begin{aligned}
\phi'_a(x) &= e^{i\Lambda(x)}\phi_a(x) \ (a=1,\cdots,N), \\
A'_\mu(x) &= A_\mu(x) - \sqrt{N}\partial_\mu\Lambda(x), \\
\sigma'(x) &= \sigma(x). \qquad \text{(D.8)}
\end{aligned}
$$

We can perform the integration over $\phi,\bar{\phi}$ to obtain

$$
\begin{aligned}
Z[J,\bar{J},J_\mu] &:= \int\mathcal{D}\sigma\int\mathcal{D}A_\mu\exp\Big\{-S_{eff} \\
&+ \int d^2x\left[\bar{J}\Delta_B^{-1}J+\frac{1}{\sqrt{N}}A_\mu J_\mu-\frac{g_0^2}{4N}J_\mu J_\mu\right]\Big\}, \qquad \text{(D.9)}
\end{aligned}
$$

where

$$
S_{eff} := N\mathrm{Tr}\ln\Delta_B + \frac{i\sqrt{N}}{g_0^2}\int d^2x\sigma(x). \qquad \text{(D.10)}
$$

The effective action is expanded in a power series of $1/N$,

$$
S_{eff} = \sum_{n=1}^{\infty} N^{1-n/2}S^{(n)} = \sqrt{N}S^{(1)} + N^0S^{(2)} + N^{-1/2}S^{(3)} + \cdots. \qquad \text{(D.11)}
$$

The diagramatic representation is given in Fig. 13 based on the rule given in Fig. 12.

First, the order $N^{1/2}$ term corresponds to the diagrams (a),(b) in Fig. 13,

$$
S^{(1)} = \frac{i}{g_0^2}\int d^2x\sigma(x) - i\mathrm{Tr}[(-\partial^2+m^2)^{-1}\sigma] \qquad \text{(D.12)}
$$

$$
= i\tilde{\sigma}(0)\left[\frac{1}{g_0^2}-\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m^2}\right], \qquad \text{(D.13)}
$$

[20] Of course, it is possible to adopt different normalizations, e.g, without explicitly introducing the factor N. Anyway, the results in the large N expansion are unchanged even if we adopt different normalizations.

where we have used the Fourier transformation,

$$\tilde{\sigma}(p) = \int d^2x e^{-ipx} \sigma(x). \tag{D.14}$$

The integral in (D.13) is ultraviolet divergent. It can be regularized by introducing the cutoff Λ. The saddle point condition $S^{(1)} = 0$ requires the bare coupling constant g_0 vary with the cutoff Λ according to

$$\frac{1}{g_0^2(\Lambda)} = \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2}. \tag{D.15}$$

In other words, if the bare coupling g_0 vary with respect to the cutoff according to

$$\frac{1}{g_0^2(\Lambda)} - \frac{1}{g_R^2(\mu)} = \frac{1}{4\pi} \ln \frac{\Lambda^2}{\mu^2}. \tag{D.16}$$

the divergences cancel in $S^{(1)}$. This implies the asymptotic freedom of CP^{N-1} model. Imposing the condition, $S^{(1)} = 0$, therefore, we obtain

$$m^2 = \mu^2 \exp\left[-\frac{4\pi}{g_R^2(\mu)} \right]. \tag{D.17}$$

Next, the order N^0 term corresponds to the diagrams (c),(d),(e) in Fig. 13,

$$S^{(2)} = \frac{1}{2} \int d^2x \int d^2y [\sigma(x)\Gamma(x,y)\sigma(y) + A_\mu(x)\Gamma_{\mu\nu}(x,y)A_\nu(y)], \tag{D.18}$$

where the Fourier transformation of $\Gamma(x,y)$ and $\Gamma_{\mu\nu}(x,y)$ are respectively given by (see Fig. 13)

$$\begin{aligned} \tilde{\Gamma}(p) &= \int \frac{d^2q}{(2\pi)^2} \frac{1}{(p^2+m^2)((p+q)^2+m^2)} && \text{(D.19)} \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{p^2(p^2+4m^2)}} \ln \frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}, && \text{(D.20)} \end{aligned}$$

and

$$\begin{aligned} \tilde{\Gamma}_{\mu\nu}(p) &= 2\delta_{\mu\nu} \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2+m^2} - \int \frac{d^2q}{(2\pi)^2} \frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{(p^2+m^2)((p+q)^2+m^2)} && \text{(D.21)} \\ &= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \left[(p^2+4m^2)\tilde{\Gamma}(p) - \frac{1}{\pi} \right]. && \text{(D.22)} \end{aligned}$$

In the neighbourhood of $p^2 = 0$,

$$\tilde{\Gamma}(p) = \frac{1}{4\pi m^2} - \frac{p^2}{24\pi m^4} + O(p^4), \quad \tilde{\Gamma}_{\mu\nu}(p) = \left[\frac{p^2}{12\pi m^2} + O(p^4) \right] \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \tag{D.23}$$

Thus we obtain the low-energy effective action,

$$S^{(2)} \cong \int d^2x \frac{1}{24\pi m^4}\sigma(x)\left(\partial^2+6m^2\right)\sigma(x) + \int d^2x \frac{1}{48\pi m^2}[\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)]^2 \tag{D.24}$$

It is important to notice that the kinetic terms for the Lagrangian multiplier fields are generated. The field σ becomes massive, while the field $A_\mu(=\sqrt{N}V_\mu)$ is massless.

The line integral in the Wilson loop is rewritten as

$$\oint_C d\xi_\mu V_\mu(\xi) = \int d^2z V_\mu(z)J_\mu(z) =: (V_\mu, J_\mu), \tag{D.25}$$

where

$$J_\mu(z) = \oint_C d\xi_\mu \delta^2(z-\xi) = \epsilon_{\mu\nu}\partial_\nu\Phi(z), \quad \Phi(z) = \begin{cases} 1 & (z \in S_C) \\ 0 & (z \notin S_C) \end{cases}, \tag{D.26}$$

where S_C is the area bounded by the loop C. Up to the leading order, we can perform the Gaussian integration to obtain

$$\begin{aligned} \left\langle \exp\left[i\oint_C d\xi_\mu V_\mu(\xi)\right]\right\rangle_{CP^{N-1}} &\cong Z^{-1}[0,0,0]\int \mathcal{D}\sigma\mathcal{D}A_\mu e^{-S^{(2)}} \exp\left(\frac{i}{\sqrt{N}}(A_\mu, J_\mu)\right) \\ &= \text{const.}\exp\left[-\frac{1}{2}\frac{12\pi m^2}{N}(J_\mu, \Delta^{-1}J_\mu)\right]. \end{aligned} \tag{D.27}$$

Thus we obtain the area law,

$$\left\langle \exp\left[i\oint_C d\xi_\mu V_\mu(\xi)\right]\right\rangle_{CP^{N-1}} = \text{const.}\exp\left[-\frac{6\pi m^2}{N}|S_C|\right], \tag{D.28}$$

since $J = {}^*d\Phi$, $\Delta := d\delta + \delta d$ and hence

$$(J_\mu, \Delta^{-1}J_\mu) := \int d^2x \int d^2y J_\mu(x)\Delta^{-1}(x,y)J_\mu(y) = (\Phi, \Phi) = |S_C|. \tag{D.29}$$

The dynamically generated gauge field A_μ produces long-range force with the linear potential which confines the ϕs. Both global $SU(N)$ and local $U(1)$ symmetries can not be broken in two dimensions due to Coleman theorem. In dimensions $D > 2$ [69], there is a critical point g_c such that for $g < g_c$ $SU(N)$ and $U(1)$ symmetries are broken, $\langle\phi_a\rangle \neq 0$, while $\langle\sigma\rangle = 0$. In this phase ϕ is regarded as the Nambu-Goldstone particle, since $m_\phi = 0$. The massless vector pole exists in the propagator $\langle A_\mu(x)A_\nu(0)\rangle$ [70]. For $g > g_c$, on the other hand, $SU(N)$ and $U(1)$ symmetries are exact, $\langle\phi_a\rangle = 0$, and $\langle\sigma\rangle \neq 0$. In this phase ϕ is massive, $m_\phi = \langle\sigma\rangle$. For $D = 2$, $g_c = 0$. For more details on the large N results, see [65, 66, 67, 68].

E Large N estimation

If we write $n^A(x)$ by the CP^{N-1} variable $\phi_a(x)$,

$$n^A(x) = \bar{\phi}_a(x)(T^A)_{ab}\phi_b(x), \tag{E.1}$$

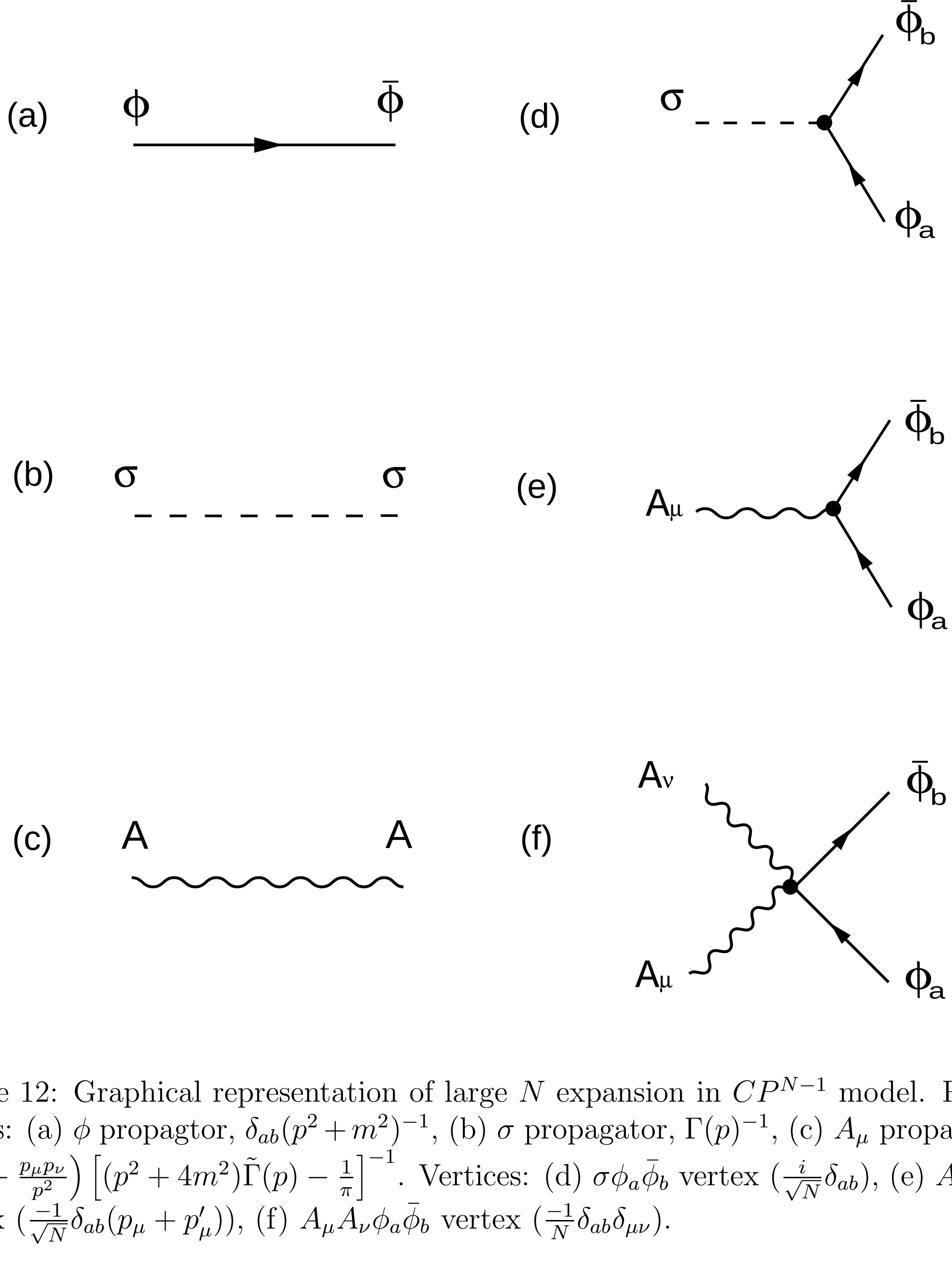


Figure 12: Graphical representation of large N expansion in CP^{N-1} model. Propagators: (a) ϕ propagtor, $\delta_{ab}(p^2+m^2)^{-1}$, (b) σ propagator, $\Gamma(p)^{-1}$, (c) A_μ propagator, $\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\left[(p^2+4m^2)\tilde{\Gamma}(p)-\frac{1}{\pi}\right]^{-1}$. Vertices: (d) $\sigma\phi_a\bar{\phi}_b$ vertex ($\frac{i}{\sqrt{N}}\delta_{ab}$), (e) $A_\mu\phi_a\bar{\phi}_b$ vertex ($\frac{-1}{\sqrt{N}}\delta_{ab}(p_\mu+p'_\mu)$), (f) $A_\mu A_\nu\phi_a\bar{\phi}_b$ vertex ($\frac{-1}{N}\delta_{ab}\delta_{\mu\nu}$).

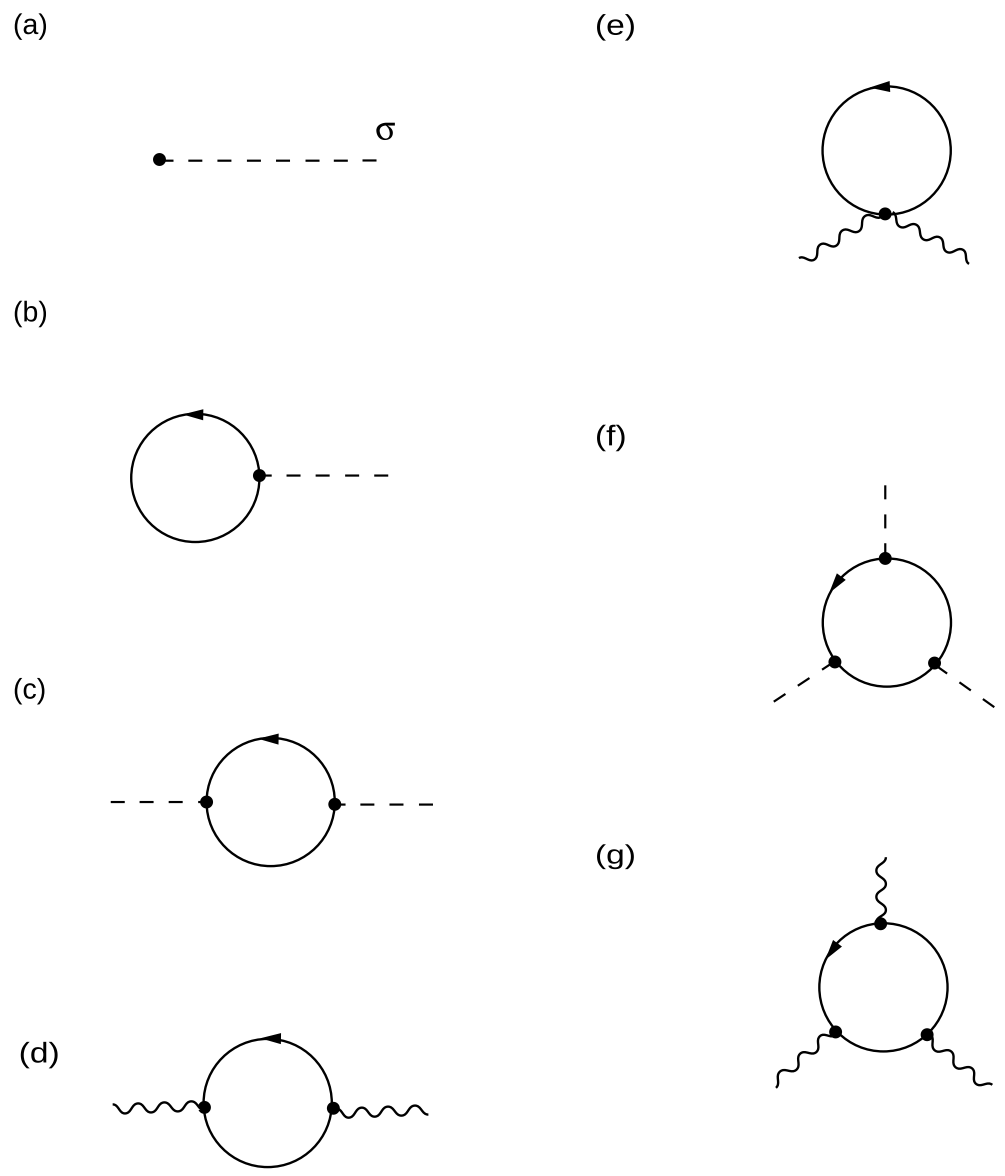


Figure 13: Examples of Feynmann diagrams. (a), (b): tadpole diagrams of order $N^{1/2}$, (c), (d), (e): vacuum polarization diagrams of order N^0. (f), (g): order $N^{-1/2}$ diagrams.

we obtain

$$
\begin{aligned}
n^A(x)n^A(y) &= \bar{\phi}_a(x)(T^A)_{ab}\phi_b(x)\bar{\phi}_c(y)(T^A)_{cd}\phi_d(y) \\
&= P_{ab}(x)P_{cd}(y)(T^A)_{ab}(T^A)_{cd} \\
&= \frac{1}{2}\left[(\bar{\phi}(x)\cdot\phi(y))(\phi(x)\cdot\bar{\phi}(y)) - \frac{1}{N}(\bar{\phi}(x)\cdot\phi(x))(\bar{\phi}(y)\cdot\phi(y))\right], \qquad \text{(E.2)}
\end{aligned}
$$

where we have used

$$
\sum_{A=1}^{N^2-1}(T^A)_{ab}(T^A)_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right). \qquad \text{(E.3)}
$$

Hence we obtain

$$
\mathbf{n}(x)\cdot\mathbf{n}(x) = n^A(x)n^A(x) = \frac{1}{2}\left[1 - \frac{1}{N}\right](\bar{\phi}(x)\cdot\phi(x))^2. \qquad \text{(E.4)}
$$

The contraint $\bar{\phi}(x)\cdot\phi(x) = 1$ leads to $\mathbf{n}(x)\cdot\mathbf{n}(x) = \frac{1}{2}\left[1 - \frac{1}{N}\right]$. The expectation value reads

$$
2\langle n^A(x)n^A(y)\rangle = \langle(\bar{\phi}(x)\cdot\phi(y))(\phi(x)\cdot\bar{\phi}(y))\rangle - \frac{1}{N}\langle|\phi(x)|^2|\phi(y)|^2\rangle. \qquad \text{(E.5)}
$$

The factorization in the large N expansion leads to

$$
\begin{aligned}
2\langle n^A(x)n^A(y)\rangle &\cong \langle\bar{\phi}(x)\cdot\phi(y)\rangle\langle\phi(x)\cdot\bar{\phi}(y)\rangle - \frac{1}{N}\langle|\phi(x)|^2\rangle\langle|\phi(y)|^2\rangle, \\
&= |\langle\bar{\phi}(x)\cdot\phi(y)\rangle|^2 - \frac{1}{N}\langle|\phi(x)|^2\rangle\langle|\phi(y)|^2\rangle. \qquad \text{(E.6)}
\end{aligned}
$$

The large N expansion shows that the field ϕ becomes massive, so that the two-point function $\langle\bar{\phi}(x)\cdot\phi(y)\rangle$ exhibits exponential decay.

In the leading order of the large N expansion, the correlation function $G_C(x,y)$ defined by (9.32) in CP_2^{N-1} model reads

$$
G_C(x,y) \cong \tilde{G}(x,y) := \left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right)\langle P_{ab}(x)P_{ba}(y)\rangle_{CP_2^{N-1}}. \qquad \text{(E.7)}
$$

Note that P is a projection operator with the properties,

$$
P^2(x) = P(x), \quad \mathrm{tr}P(x) = 1. \qquad \text{(E.8)}
$$

The composite operator P is $U(1)$ gauge invariant. SU(N) invariance implies that

$$
\langle P_{ab}(x)\rangle = \langle\bar{\phi}_a(x)\phi_b(x)\rangle = \frac{1}{N}\delta_{ab}. \qquad \text{(E.9)}
$$

It is easy to show that

$$
\begin{aligned}
\tilde{G}(x,y) &= \langle\bar{\phi}_a(x)\phi_b(x)\bar{\phi}_b(y)\phi_a(y)\rangle - \frac{1}{N} \\
&= \langle D^2(x,y)\rangle + \frac{1}{N}\langle D(x,x)D(y,y)\rangle - \frac{1}{N} \\
&= \langle D^2(x,y)\rangle + \frac{1}{N}\langle D(x,x)D(y,y)\rangle_{conn}, \qquad \text{(E.10)}
\end{aligned}
$$

where $\langle D(x,x)\rangle = 1$.

Acknowledgments

This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.10640249).

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